Econometrics

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2009

Convergence in distribution

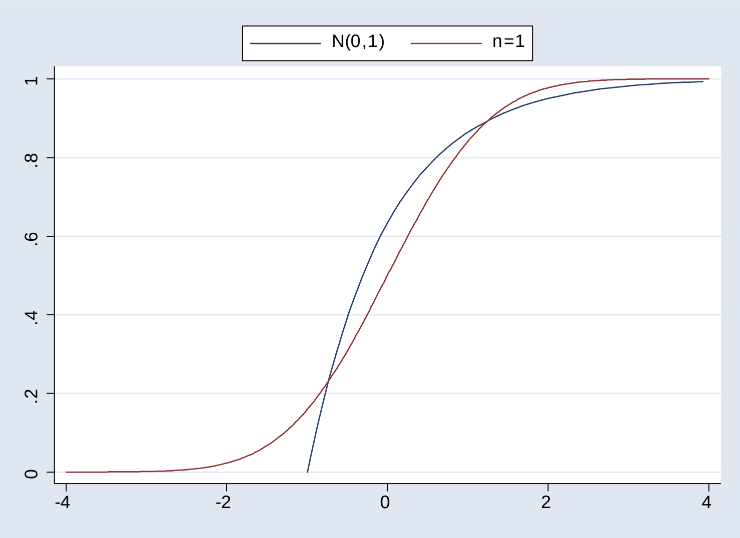
- **Formal definition**: $\mathbf{a}_n \xrightarrow{D} \mathbf{a}$, if for every $\mathbf{c}$, for which distribution F is continuos:

$$\lim_{n \longrightarrow \infty} F_n(\mathbf{c}) = F(\mathbf{c}),$$

  where $F_n(\bullet)$ is distribution of $\mathbf{a}_n$ and $F(\bullet)$ is distribution of $\mathbf{a}$
- **Informal definition:** sequence of random variables $\mathbf{a}_n$ converge in distribution to $F(\bullet)$ if for large n distribution of $\mathbf{a}_n$ become "very close" to distribution given by $F(\bullet)$.

Convergence in distribution

Distribution of random variable $\sqrt{n}\frac{(\overline{x}-\mu)}{\sigma^2}$, where $x_i \sim \chi^2_2$



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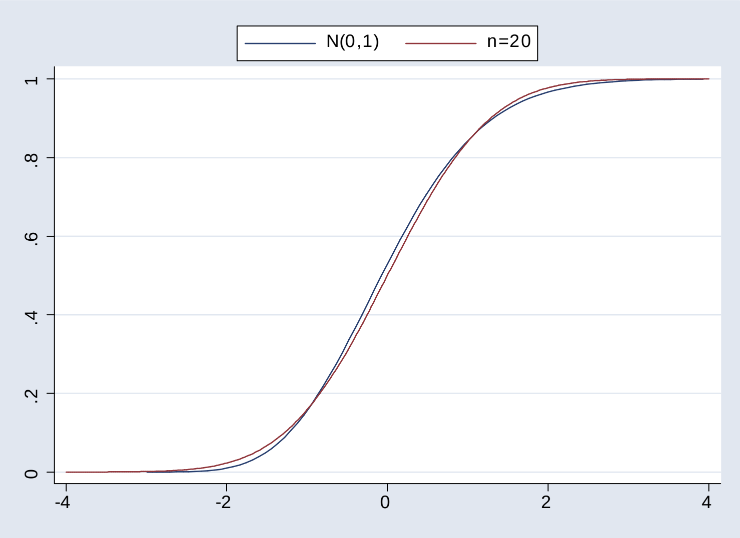


RESET test

- *RESET:* **Re***gression* **S***pecification* **E***rror* **T***est*
- Null and alternative hypothesis

$$\begin{aligned} H_0 &: \quad y_i = \mathbf{x}_i \boldsymbol{\beta} + \varepsilon_i \\ H_1 &: \quad y_i = f\left(\mathbf{x}_i\right) + \varepsilon_i \end{aligned}$$

- Auxiliary regression

$$y_i = \mathbf{x}_i \widehat{\boldsymbol{\beta}} + \alpha_1 \widehat{y}_i^2 + \ldots + \alpha_p \widehat{y}_i^{p+1} + \widehat{u}_i$$

- Test statistics F for $H_0 : \boldsymbol{\alpha} = \mathbf{0}$.
- Second form:

$$e_i = \mathbf{x}_i \widehat{\boldsymbol{\beta}} + \widehat{\mathbf{z}}_i \widehat{\boldsymbol{\alpha}} + \widehat{u}_i$$

- Test statistics $LM = NR^2 \xrightarrow{D} \chi_p^2$ where $\widehat{\mathbf{z}}_i = \left(\widehat{y}_i^2, \ldots, \alpha_p \widehat{y}_i^{p+1}\right)$
- Interpretation of the test result:
  - H_0 rejected $\Longrightarrow$functional for of the model is invalid
- Consequences of the rejection of $H_0:$ if the functional form is invalid then the for of the dependence between y_i, $\mathbf{x}_i$ and $\boldsymbol{\beta}$ is also unknown

Jarque-Bera Test

- Null and alternative hypothesis:

$$
\begin{aligned}
H_0 &: \varepsilon \sim N\left(0, \sigma^2 \mathbf{I}\right) \\
H_0 &: \varepsilon \nsim N\left(0, \sigma^2 \mathbf{I}\right)
\end{aligned}
$$

- Test statistics

$$LM = N\left[\frac{\widehat{\theta}_1^2}{6} + \frac{\left(\widehat{\theta}_2 - 3\right)^2}{24}\right] \xrightarrow{D} \chi_2^2$$

where

$$
\begin{aligned}
\widehat{\theta}_1 &= \frac{\sum_{i=1}^n e_i^3 / n}{\widehat{\sigma}^3} \quad \text{skewness} \\
\widehat{\theta}_2 &= \frac{\sum_{i=1}^n e_i^4 / n}{\widehat{\sigma}^4} \quad \text{kurtosis} \\
\widehat{\sigma} &= \sqrt{\frac{\sum_{i=1}^n e_i^2}{n}} \quad \text{standard error}
\end{aligned}
$$

Jarque-Bera Test

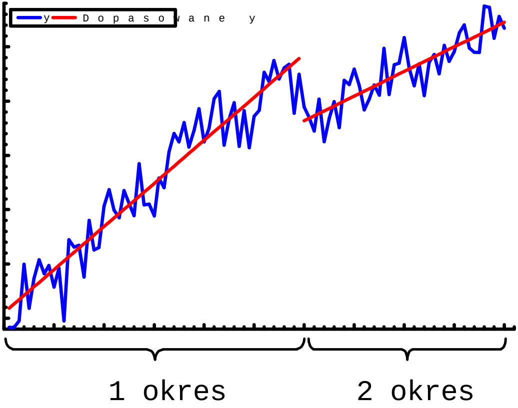
- Interpretation of the test result:
  - H_0 rejected $\Longrightarrow$ distribution of the error term is not normal
- Consequences of the rejection of H_0 : assumption of normality of error terms is used when we derive the small sample distribution of estimators and tests statistics. If this assumption is invalid we can only use the asymptotic distributions.

Chow test

- Null and alternative hypothesis

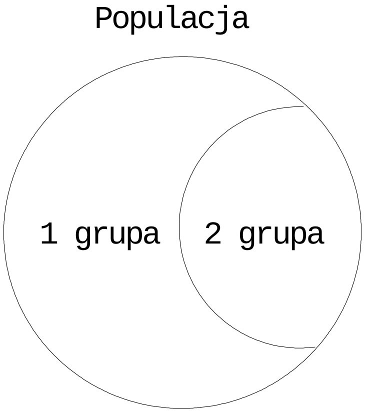
$$
\begin{aligned}
H_0 &: \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_m \\
H_1 &: \boldsymbol{\beta}_r \neq \boldsymbol{\beta}_s \text{ for some } r, s
\end{aligned}
$$

At time t^* parameters of the model are changing (structural change in the economy



Chow test
Cross section

Subsample of the sample should be described with a different model than the rest of sample



- Estimated model

$$y_i = \sum_{s=1}^{m} Q_{s,i} \mathbf{x}_i \boldsymbol{\beta}_s + \varepsilon_i, \quad \boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \sigma^2 \mathbf{I}_N\right)$$

$$Q_{s,i} = \begin{cases} 1 & \text{if observation } i \text{ belongs to subsample } s \\ 0 & \text{if observation } i \text{ does not belong to subsample } s \end{cases}$$

- We test for $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_m$
- The form of the test statistics:

$$F = \frac{\left(S_R - \sum_i S_i\right) / \left[K\left(m-1\right)\right]}{\sum_i S_i / \left(N - mK\right)} \sim F_{K(m-1),N-mK}$$

  where S_i is the residual sum of squares for model estimated on subsample i and S_R is the residual sum of squares for model estimated on the full sample
- Interpretation of the test result:
  - H_0 rejected $\Longrightarrow$ parameters of the model are not stable
- Consequences of the rejection of H_0: model should not be estimated on full sample. However, it is often possible to find subsample for which parameters are stable.

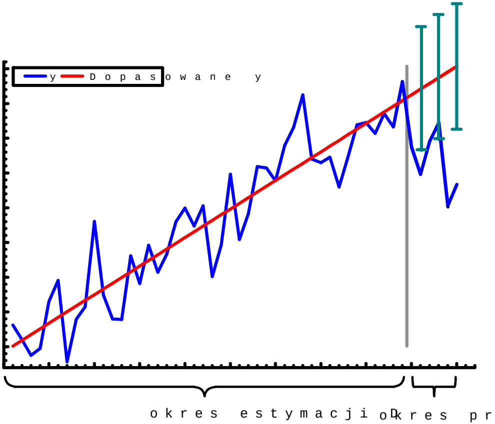
Forecast test

- Null and alternative hypothesis

$$\begin{aligned} H_0 &: \mathbf{y}_F = \mathbf{X}_F \boldsymbol{\beta} + \varepsilon_F \\ H_1 &: \mathbf{y}_F \neq \mathbf{X}_F \boldsymbol{\beta} + \varepsilon_F \end{aligned}$$

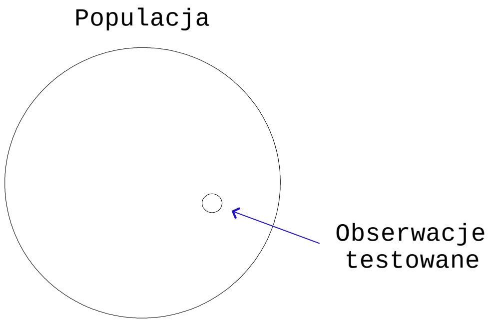
Forecast test

time series



Forecast test

cross section



- Estimation and forecast period:

$$\underbrace{1,\ldots,n}_{\text{estimation period}} \qquad \underbrace{n+1,\ldots,n+g}_{\text{forecast period}}$$

- Statistics:

$$F = \frac{\left(S_{E+F} - S_E\right)/\,g}{S_E/\left(N-K\right)} \sim F\left(g, N-K\right)$$

where S_{E+F} is residual sum of squares for the model estimated on full sample and S_E is residual sum of squares for the model estimated only on the data from estimation period

- Interpretation of the test result:
  - H_0 rejected $\Longrightarrow$ we reject the hypothesis, that the observations $n+1,\ldots,n+g$ are coming from the same model as observations for $1,\ldots,n$

- Consequences of the rejection of H_0: is is not possible to generalize the results of estimation for $1,\ldots,n$ for observations $n+1,\ldots,n+g$

Goldfelda-Quandt test

- Null and alternative hypothesis

$$
\begin{aligned}
H_0 &: \mathrm{Var}\left(\varepsilon_i\right) = \sigma^2 && \text{for } i = 1, \ldots, N \\
H_1 &: \mathrm{Var}\left(\varepsilon_i\right) > \mathrm{Var}\left(\varepsilon_j\right) && \text{for } z_i > z_j
\end{aligned}
$$

Goldfelda-Quandt test

test procedure

	sort in descending order according to z_i	observations	
	z_1	$\mathbf{x}_1$	y_1
$\downarrow$	$\vdots$	$\vdots$	$\vdots$
	z_{n_1}	$\mathbf{x}_{n_1}$	y_{n_1}
	z_{n_1+1}	$\mathbf{x}_{n_1+1}$	y_{n_1+1}
$\downarrow$	$\vdots$	$\vdots$	$\vdots$
	z_{n_1+c}	$\mathbf{x}_{n_1+c}$	y_{n_1+c}
	z_{n_1+c+1}	$\mathbf{x}_{n_1+c+1}$	y_{n_1+c+1}
$\downarrow$	$\vdots$	$\vdots$	$\vdots$
	$z_{n_1+n_2+c}$	$\mathbf{x}_{n_1+n_2+c}$	$y_{n_1+n_2+c}$

We regress y_t on $\mathbf{X}$ separately for group 1 and for group 2
Test statistics:

$$\frac{\mathbf{e}_1'\mathbf{e}_1 \big/ (n_1 - K)}{\mathbf{e}_2'\mathbf{e}_2 \big/ (n_2 - K)} \sim F\,(n_1 - K, n_2 - K)$$

- Interpretation of the test result:
  - H_0 rejected $\Longrightarrow$ we reject hypothesis, that the variance of error term is the same for all the observations
- Advantage of Goldfeld Quandt test: can be used for small samples
- Disadvantage: variance can only depend monotonically on one variable

Breusch-Pagan test

- Null and alternative hypothesis

$$H_0 : \mathrm{Var}\left(\varepsilon_i | \mathbf{z}_i\right) = \sigma^2 \qquad \text{for } i = 1, \ldots, N$$
$$H_1 : \mathrm{Var}\left(\varepsilon_i | \mathbf{z}_i\right) = \sigma_i^2 = \sigma^2 f\left(\alpha_0 + \mathbf{z}_i \boldsymbol{\alpha}\right)$$

- Test procedure:

1. Estimate base model
$$y_i = \mathbf{x}_i \boldsymbol{\beta} + \varepsilon_i$$
2. Estimate auxiliary regression $\frac{e_i^2}{\tilde{\sigma}^2}$ on $\mathbf{z}_i$
3. Use one of the statistics:
$$\frac{1}{2} ESS \xrightarrow{D} \chi_p^2$$
$$nR^2 \xrightarrow{D} \chi_p^2$$

where p is the number of variables included in $\mathbf{z}_i$

- test advantages: more general form of heteroskedasticity
- disadvantages: only the asymptotic distribution is known

Durbin-Watson test

- Null and alternative hypothesis

$$H_0 : \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t-1}\right) = 0$$
$$H_1 : \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t-1}\right) \neq 0$$

- We estimate base model

$$y_i = \mathbf{x}_i \boldsymbol{\beta} + \varepsilon_i$$

- On the basis of residuals from base regression we construct the statistic

Durbin-Watson test

$$
\begin{aligned}
DW &= \frac{\sum_{t=2}^{T}\left(e_t - e_{t-1}\right)^2}{\sum_{t=1}^{T} e_t^2} \\
&= \frac{2\sum_{t=1}^{T} e_t^2 - 2\sum_{t=2}^{T} e_t e_{t-1} - e_1^2 - e_T^2}{\sum_{t=1}^{T} e_t^2} \\
&= 2\left(1 - \widehat{\rho}_{\varepsilon_t,\varepsilon_{t-1}}\right) - \frac{e_1^2 + e_T^2}{\sum_{t=1}^{T} e_t^2} \\
&\xrightarrow{p} 2\left(1 - \rho_{\varepsilon_t,\varepsilon_{t-1}}\right)
\end{aligned}
$$

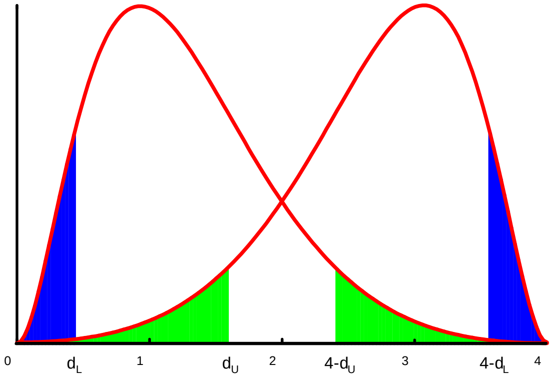
as from Law of Large Numbers we know, that $\widehat{\rho}_{\varepsilon_t,\varepsilon_{t-1}} \xrightarrow{p} \rho_{\varepsilon_t,\varepsilon_{t-1}}$ and $\frac{e_1^2+e_T^2}{\sum_{t=1}^{T} e_t^2} \xrightarrow{p} 0$.

Durbin-Watson test

1. Inference in the case of DW test looks as follows:
    1. if $DW < 2$
        1. $DW < d_L$ we reject H_0 of no autocorrelation and accept the alternative of positive autocorrelation
        2. $d_L < DW < d_U$ no conclusion
        3. $DW > d_U$ we cannot reject H_0 of no autocorrelation
    2. if $DW > 2$
        1. $DW > 4 - d_L$ we reject H_0 of no autocorrelation and accept the alternative of negative autocorrelation
        2. $4 - d_U < DW < 4 - d_L$ no conclusion
        3. $DW < 4 - d_U$ we cannot reject H_0 of no autocorrelation

- The reason why the DW statistic have no conclusion region is the dependence of the distribution of this statistics from the form of the $\mathbf{X}$ matrix.

Distribution of Durbin-Watson test



- advantages of *DW* test: small sample test
- disadvantages of *DW* test:
  - no conclusion region
  - can only detect the autocorrelation of the first order $\mathrm{E}\left[\varepsilon_t \varepsilon_{t-1}\right] \neq 0$
  - cannot be used if the lagged dependent variable is included in the mode

Breusch-Godfrey test

- Null and alternative hypothesis:

$$H_0 : \mathrm{Cov}\left(\varepsilon_t, \varepsilon_{t-i}\right) = 0 \qquad \text{where } i = 1, \ldots, s$$
$$H_0 : \varepsilon_t = \gamma_1 \varepsilon_{t-1} + \ldots + \gamma_s \varepsilon_s + u_t \quad \text{where } \mathrm{Var}\left(\mathbf{u}\right) = \sigma_u^2 \mathbf{I}$$

Breusch-Godfrey test

- Test procedure:

1. We estimate base model

$$y_i = \mathbf{x}_i \boldsymbol{\beta} + \varepsilon_i$$

2. We regress the residuals from base regression on explanatory variables and lagged residuals

$$e_t = \mathbf{x}_t \boldsymbol{\mu} + \gamma_1 e_{t-1}, \ldots, \gamma_s e_{t-s}$$

3. We test for autocorrelation by testing in auxiliary regression the hypothesis

$$H_0 : \gamma_1 = \ldots = \gamma_s = 0$$

it can be done with statistic F or the statistic

$$TR^2 \xrightarrow{D} \chi^2_s,$$

where R^2 is coming from auxiliary regression.

- test advantages: more general form of autocorrelation
- disadvantages: only the asymptotic distribution is known