Autoregressive models with distributed lags (ADL)

- It often happens than including the lagged dependent variable in the model results in model which is better fitted and needs less parameters.

- It can be explained by the inertia of many economic processes

- Model in which we have lagged explanatory variables is called autoregressive model

- General class of such a models are ADL models (**A**uto**r**egressive

Distributed **L**ags) of the form:

$$y_t = \underbrace{\boldsymbol{\alpha}_1 y_{t-1} + \ldots + \boldsymbol{\alpha}_p y_{t-p}}_{AR}$$
$$+ \underbrace{\boldsymbol{\mu} + \boldsymbol{x}_t \boldsymbol{\beta}_0 + \boldsymbol{x}_{t-1} \boldsymbol{\beta}_1 + \ldots + \boldsymbol{x}_{t-s} \boldsymbol{\beta}_s}_{DL} + \boldsymbol{\varepsilon}_t$$

Impact and long run multipliers in ADL

- Impact multiplier is the same os in DL models and is equal to

$$\frac{\mathrm{E}\left(\Delta y_t\right)}{\Delta \boldsymbol{x}_t} = \boldsymbol{\beta}_0$$

- Log run multiplier can be calculated in a following way. We change $\boldsymbol{x}_t$, $\boldsymbol{x}_{t-1}, \ldots$ by $\Delta \boldsymbol{x}$. The expected change of y_t is equal to:

$$\begin{aligned} \mathrm{E}\left(y_t + \Delta y_t\right) &= \boldsymbol{\alpha}_1 \mathrm{E}\left(y_{t-1} + \Delta y_{t-1}\right) + \ldots + \boldsymbol{\alpha}_p \mathrm{E}\left(y_{t-p} + \Delta y_{t-p}\right) \\ &\quad + \boldsymbol{\mu} + \left(\boldsymbol{x}_t + \Delta \boldsymbol{x}\right) \boldsymbol{\beta}_0 \\ &\quad + \left(\boldsymbol{x}_{t-1} + \Delta \boldsymbol{x}\right) \boldsymbol{\beta}_1 + \ldots + \left(\boldsymbol{x}_{t-s} + \Delta \boldsymbol{x}\right) \boldsymbol{\beta}_s \end{aligned}$$

and then y_t is equal to

$$\begin{aligned}\mathrm{E}(\Delta y_t) &= \boldsymbol{\alpha}_1 \mathrm{E}(\Delta y_{t-1}) + \ldots + \boldsymbol{\alpha}_p \mathrm{E}(\Delta y_{t-p}) \\ &\quad + \Delta \boldsymbol{x}(\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 + \ldots + \boldsymbol{\beta}_s)\end{aligned}$$

- But in long run the influence of permanent change of x on y stabilizes:

$$\mathrm{E}(\Delta y) = \mathrm{E}(\Delta y_t) = \mathrm{E}(\Delta y_{t-1}) = \ldots = \mathrm{E}(\Delta y_{t-p})$$

- So the long run multiplier $\boldsymbol{\beta}$ is equal to:

$$\frac{\mathrm{E}(\Delta y)}{\Delta \boldsymbol{x}} = \boldsymbol{\beta} = \frac{\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 + \ldots + \boldsymbol{\beta}_s}{1 - \boldsymbol{\alpha}_1 - \ldots - \boldsymbol{\alpha}_p}$$

Example 1. *Unemployment and money supply (cont.). We introduce the lagged variable unemployment as explanatory variable and eliminate insignificant lags of explanatory variables.*

```
      Source |       SS       df       MS              Number of obs =      42
-------------+------------------------------           F(  2,    39) =  246.00
       Model |  452.505732     2  226.252866           Prob > F      =  0.0000
    Residual |  35.8693916    39   .91972799           R-squared     =  0.9266
-------------+------------------------------           Adj R-squared =  0.9228
       Total |  488.375123    41  11.9115884           Root MSE      =  .95902

------------------------------------------------------------------------------
bael         |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
bael         |
          L1 |   .8508812   .0571166    14.90   0.000      .735352    .9664103
m3p          |  -.1570696   .0519469    -3.02   0.004    -.2621422   -.051997
_cons        |   3.015409    1.04154     2.90   0.006     .9086948    5.122124
------------------------------------------------------------------------------
```

- Obtained model is better fitted but has less parameters

- Impact multiplier of the money supply is equal to: $\boldsymbol{\beta}_0 = -.1570696$

- Long run multiplier of the money supply is equal to: $\boldsymbol{\beta} = \frac{-.1570696}{1-.8508812} = -1.0533$

- Long run multiplier is approximately the same as in the DL model!

- Breusch-Godfrey test:

```
Breusch-Godfrey LM test for autocorrelation
---------------------------------------------------------------------------
    lags(p)  |          chi2               df                 Prob > chi2
-------------+-------------------------------------------------------------
       1     |          0.309               1                   0.5782
---------------------------------------------------------------------------
                        H0: no serial correlation
```

- No autocorrelation - dynamics of unemployment seems to be well described!

Long run equilibrium

- Long run equilibrium (long run solution) it the state in which the expected value of the depend variables is constant if the exogenous variables are constant

- Long run equilibrium is interesting from the point of view of the theory because the majority of macroeconomic theories concern the relations between the economic variables in equilibrium

- The equilibrium solution we find using its definition, by assuming that the expected value of the dependent variable and independent variables are constant over time

$$
\begin{aligned}
y^* &= \mathrm{E}(y_t) = \mathrm{E}(y_{t-1}) = \ldots = \mathrm{E}(y_{t-p}) \\
\boldsymbol{x}^* &= \boldsymbol{x}_t = \boldsymbol{x}_{t-1} = \ldots = \boldsymbol{x}_{t-s}
\end{aligned}
$$

- Substituting into the definition of ADL we get:

$$(1 - \boldsymbol{\alpha}_1 - \ldots - \boldsymbol{\alpha}_p)\, y^* = \boldsymbol{\mu} + \boldsymbol{x}^*\boldsymbol{\beta}_0 + \boldsymbol{x}^*\boldsymbol{\beta}_1 + \ldots + \boldsymbol{x}^*\boldsymbol{\beta}_s$$

- For model ADL it implies that long run equilibrium is given by:

$$y^* = \boldsymbol{\mu}^* + \boldsymbol{x}^*\boldsymbol{\beta}$$

where $\boldsymbol{\mu}^* = \frac{\boldsymbol{\mu}}{1-\boldsymbol{\alpha}_1-\ldots-\boldsymbol{\alpha}_p}$ a $\boldsymbol{\beta} = \frac{\boldsymbol{\beta}_0+\boldsymbol{\beta}_1+\ldots+\boldsymbol{\beta}_s}{1-\boldsymbol{\alpha}_1-\ldots-\boldsymbol{\alpha}_p}$.

- Notice that β is exactly equal to long run multiplier

Example 2. *Unemployment and money supply (cont.). The long run equilibrium in this model is given by:*

$$
\begin{aligned}
y^* &= \frac{3.015409}{1-.8508812} + \frac{-.1570696}{1-.8508812}x^* \\
&= 20.222 - 1.0533x^*
\end{aligned}
$$

For money supply expansion equal to zero, the unemployment would be equal to 20.222%

Causality testing

- How the cause and the result can be defined?

- Features of causality:

  - cause always is precede the result
  - if the cause take place we can predict that the result will take place as well

- Granger causality (very special definition of causality cannot be taken as definition of causality in philosophical sense)

Definition 3. *Variable x is Granger cause of y if current values of y can be better forecasted with the use of past x's than without them.*

- In model

$$y_t = a(t) + \sum_{i=1}^{k} \boldsymbol{\alpha}_i y_{t-i} + \sum_{i=1}^{k} \boldsymbol{\beta}_i x_{t-i} + \boldsymbol{\varepsilon}_t$$

  where $a(t)$ is deterministic part of the model (e.g. $a(t) = \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1 t$), if $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_k = 0$, then x is not a Granger cause of y

- **Granger causality test**:

- We test the null hypothesis that x *is not a Granger cause of* y by testing joint hypothesis that all $\boldsymbol{\beta}_i$ related to lagged x are equal to zero $H_0 : \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \ldots = \boldsymbol{\beta}_k = 0$

Example 4. *Unemployment and money supply (cont.). In order to test Granger causality we should build the model with only lagged values of the variable the causality of which we are testing (contemporaneous x's should not be included)*

```
      Source |       SS       df       MS              Number of obs =      42
-------------+------------------------------           F(  2,    39) =  200.84
       Model |  445.153629     2  222.576814           Prob > F      =  0.0000
    Residual |  43.2214946    39  1.10824345           R-squared     =  0.9115
-------------+------------------------------           Adj R-squared =  0.9070
       Total |  488.375123    41  11.9115884           Root MSE      =  1.0527

------------------------------------------------------------------------------
bael         |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
bael         |
          L1 |   1.007958   .0673853    14.96   0.000     .8716585    1.144258
m3p          |
          L1 |   .0603557    .061816     0.98   0.335    -.0646789    .1853903
_cons        |  -.3409437   1.249234    -0.27   0.786    -2.867759    2.185871
------------------------------------------------------------------------------
```

- In this simple model we any need to test the significance of lagged $m3p$
- According to test result, changes of money supply are not Granger cause of changes in unemployment!

- Knowledge of the changes of money supply in the past cannot be used for improving the forecasts of the future unemployment.

Consistency of OLS in estimation of ADL

- Variables and ADL model should be predetermined (exogenous or lagged endogenous)
- If there is an autocorrelation in the model with lagged depend variables, then we simultaneity will occur
- In this case the $\boldsymbol{b}_{OLS}$ will not be consistent
- Because of this problem we should always try to eliminate the autocorrelation from the model wit autoregressive part
- For testing autocorrelation we can use Breusch-Godfrey test

- In model with autoregressive part the DW test *cannot be used* for testing autocorrelation as it is consistent

- Autocorrelation can normally be eliminated by adding lags of dependent variable to the model

Lag operator

- We define L as

$$x_{t-1} = Lx_t$$

- As

$$LLx_t = Lx_{t-1} = x_{t-2} = L^2 x_t,$$

the s power of L can be defined as such operator that

$$L^s x_t = x_{t-s}$$

- As

$$\left[1 + aL + (aL)^2 + \ldots (aL)^n\right](1 - aL) = 1 - (aL)^{n+1}$$

then

$$1 + aL + (aL)^2 + \ldots (aL)^n = (1 - aL)^{-1} \left[1 - (aL)^{n+1}\right]$$

- If $|a| < 1$ then

$$(1 - aL)^{-1} = \sum_{i=1}^{\infty} (aL)^i$$

- Conclusion: $1 - aL$ can be inverted, if $|a| < 1$

Complex numbers and Fundamental Theorem of Algebra

- Complex number x can be represented as

$$x = a + bi$$

where a and b are real numbers and $i = \sqrt{-1}$. Obviously $i^2 = -1$.

- Standard proporties of summation and multiplication apply to complex numbers

- Modulus of complex number is defined as $|x| = \sqrt{a^2 + b^2}$

- Fundamental theorem of algebra: polynomial

$$A(x) = 1 - a_1 x - a_2 x^2 - \ldots - a_s x^s$$

can be factorized into

$$A(x) = (1 - \boldsymbol{\lambda}_1 x) \times \ldots \times (1 - \boldsymbol{\lambda}_s x)$$

where $\boldsymbol{\mu}_i = \boldsymbol{\lambda}_i^{-1}$ are roots (possibly complex) of this polynomial.

Polynomials of lag operator

- Polynomial of lag operator:

$$A(L) = 1 - a_1 L - a_2 L^2 - \ldots - a_s L^s$$

- Fundamental theorem of algebra implies that the polynomial of lag operator can be factorized as follows

$$A(L) = (1 - \boldsymbol{\lambda}_1 L) \times \ldots \times (1 - \boldsymbol{\lambda}_s L)$$

- $1 - \boldsymbol{\lambda}_i L$ can be inverted if $|\boldsymbol{\lambda}_i| < 1$ (modulus of complex number is smaller than 1).

- $A(L)$ can only be inverted if $|\boldsymbol{\lambda}_i| < 1$ (or $|\boldsymbol{\mu}_i| > 1$) for $i = 1, \ldots, s$

- $A(L)$ can be inverted only if all its roots lie *outside* the unit circle

- The $A(L)^{-1}$ can the be written as

$$
\begin{aligned}
A(L)^{-1} &= (1 - \boldsymbol{\lambda}_1 L)^{-1} \times \ldots \times (1 - \boldsymbol{\lambda}_s L)^{-1} \\
&= \left[\sum_{i=1}^{\infty} (\boldsymbol{\lambda}_1 L)^i\right] \times \ldots \times \left[\sum_{i=1}^{\infty} (\boldsymbol{\lambda}_s L)^i\right] \\
&= \sum_{i=0}^{\infty} \boldsymbol{\psi}_i L^i
\end{aligned}
$$

where $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots$ are parameters converging to zero.

Example 5. ADL *model written with the use of polynomials of lag operator:*

$$y_t - \boldsymbol{\alpha}_1 y_{t-1} - \ldots - \boldsymbol{\alpha}_p y_{t-p} = \boldsymbol{\mu} + \boldsymbol{x}_t \boldsymbol{\beta}_0 + \boldsymbol{x}_{t-1} \boldsymbol{\beta}_1 + \ldots + \boldsymbol{x}_{t-s} \boldsymbol{\beta}_s + \boldsymbol{\varepsilon}_t$$

$$A(L)\, y_t = \boldsymbol{\mu} + B(L)\, \boldsymbol{x}_t + \boldsymbol{\varepsilon}_t$$

where

$$\begin{aligned} A(L) &= 1 - \boldsymbol{\alpha}_1 L - \ldots - \boldsymbol{\alpha}_p L^p \\ B(L) &= \boldsymbol{\beta}_0 - \boldsymbol{\beta}_1 L - \ldots - \boldsymbol{\beta}_s L^s \end{aligned}$$

If all the root of polynomial $A(L)$ *lie outside the unit circle we can write*

$$\begin{aligned} y_t &= \boldsymbol{\mu} + A(L)^{-1} B(L)\, \boldsymbol{x}_t + A(L)^{-1} \boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\mu}^* + \sum_{i=0}^{\infty} \boldsymbol{\psi}_i \boldsymbol{x}_{t-i} + \sum_{i=0}^{\infty} \boldsymbol{\phi}_i \boldsymbol{\varepsilon}_{t-i} \end{aligned}$$

ADL model is equivalent to DL model with infinite number of lags and autocorrelated (moving average) error term!

Difference operator

- Definition

$$\Delta \boldsymbol{x}_t = \boldsymbol{x}_t - \boldsymbol{x}_{t-1} = (1 - L)\,\boldsymbol{x}_t$$

so $\Delta \equiv 1 - L$.

- Differences of order p we denote as

$$\Delta^p = (1 - L)^p$$

- For instance

$$\begin{aligned}\Delta^2 \boldsymbol{x}_t &= (1 - L)^2\,\boldsymbol{x}_t = \left(1 - 2L + L^2\right)\boldsymbol{x}_t \\ &= \boldsymbol{x}_t - 2x_{t-1} + \boldsymbol{x}_{t-2} = \Delta \boldsymbol{x}_t - \Delta \boldsymbol{x}_{t-1}\end{aligned}$$

- Seasonal differences:

$$\Delta_s = 1 - L^s$$

- For quarterly data:

$$\Delta_4 x_t = x_t - x_{t-4}$$

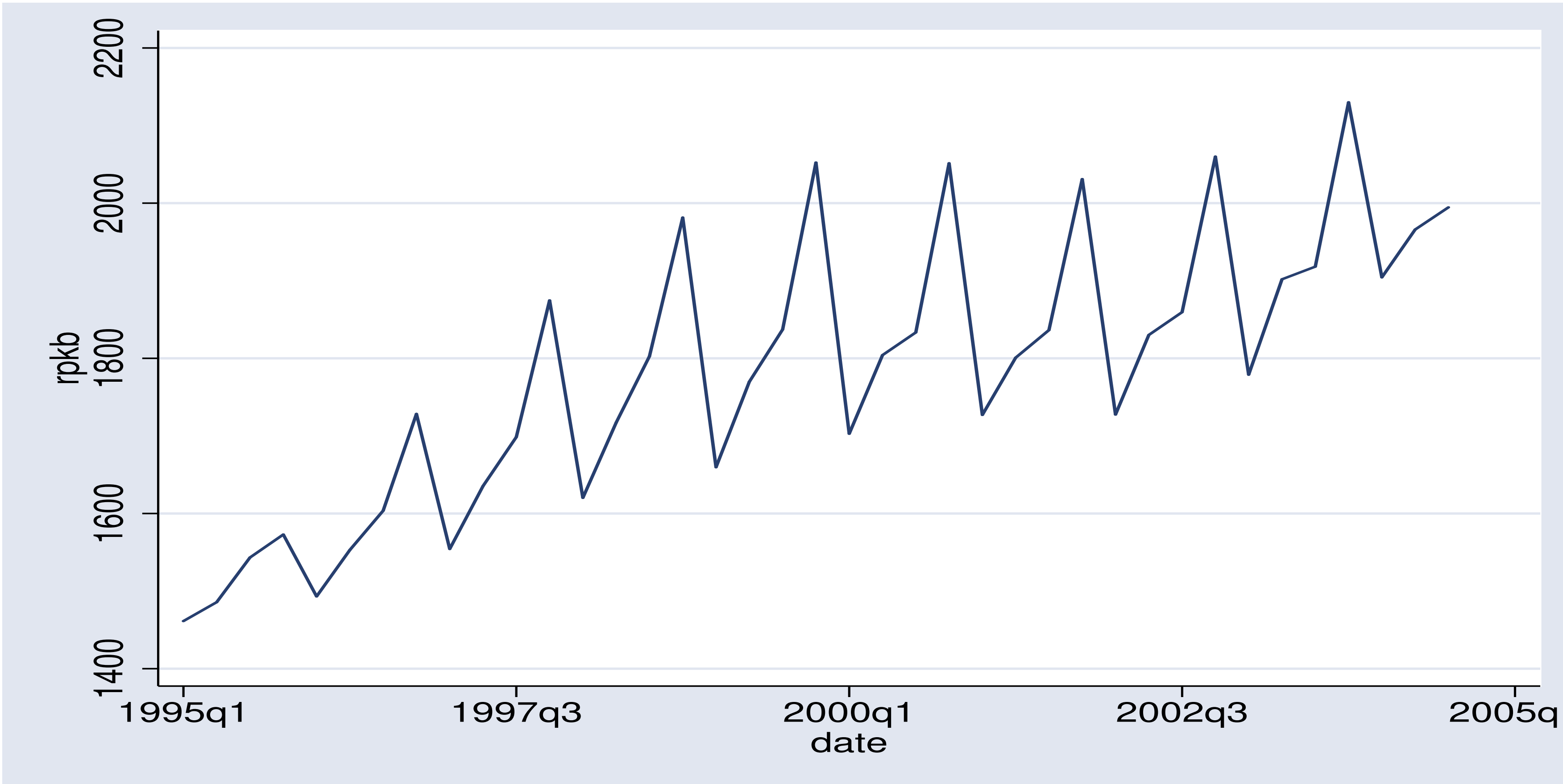
Seasonality

- Seasonality take place if we observe cyclical changes in data related to seasons of the year

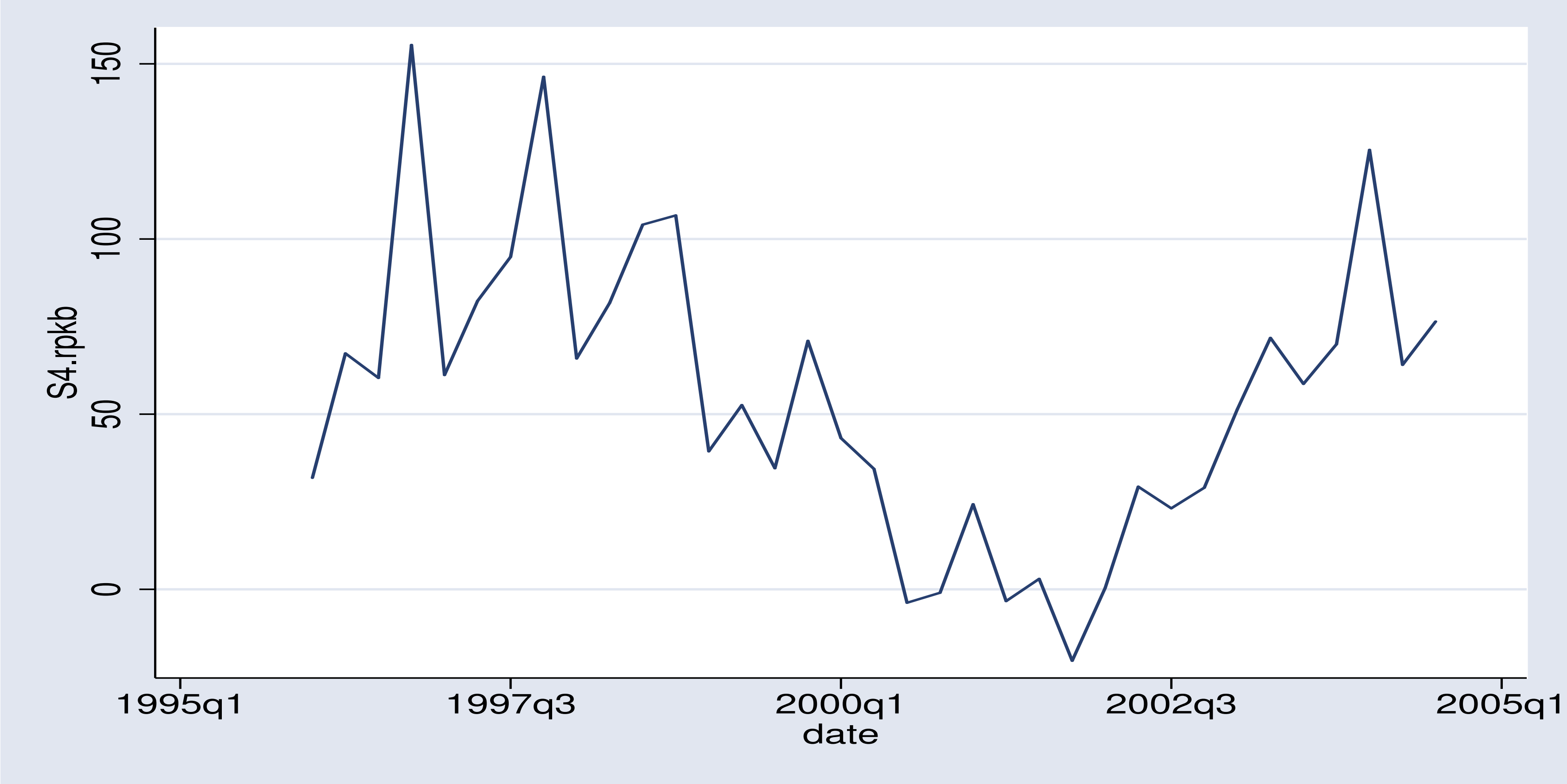
- For example quarterly data often shows quarterly seasonality and monthly data monthly seasonality.

- Seasonality in the model can be taken into account into model in two ways:
  - we can include in the model additional dummy variables related to each quarter (month)
  - we can use seasonal differencing: for example in the case of quarterly data $\Delta_4 y_t = y_t - y_{t-4}$

Example 6. *GDP in Poland in years 1996-2004. Seasonal changes in this case are related to investment cycle - the largest investments are registered in forth quarter.*



- PKB in real terms - CPI deflator



- PKB in real terms, seasonally differenced