Autoregressive conditional heteroskedasticity

Jerzy Mycielski

Uniwersytet Warszawski

May2019

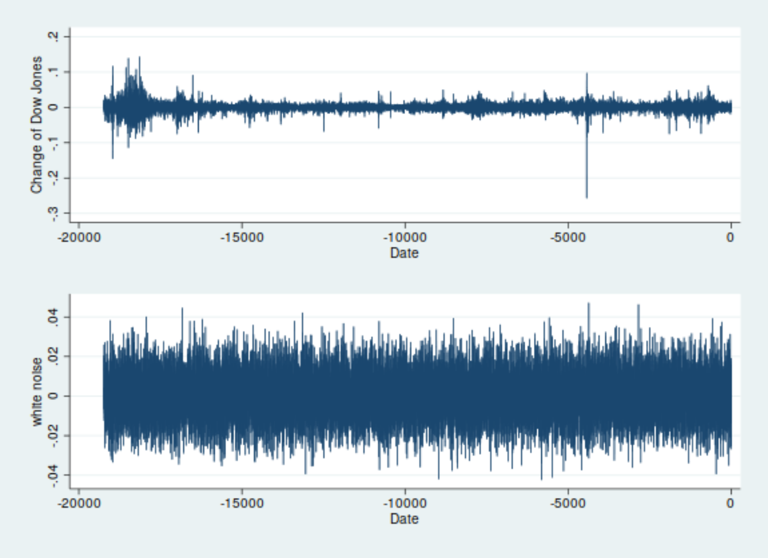
ARCH(1)

- $ARCH(1)$ Model (**A**uto**r**egressive **C**onditional **H**eteroskedasticity)

$$\begin{aligned} y_t &= \mathbf{x}_t \beta + \varepsilon_t \\ \varepsilon_t &= u_t \sqrt{\theta_0 + \theta_1 \varepsilon_{t-1}^2} \end{aligned}$$

where $u_t \sim N(0,1)$, u_t is independent from ε_{t-1} and $E(\varepsilon_t | \mathbf{x}_t, \varepsilon_{t-1}) = 0$

Dow Jones



Change of Dow Jones
.2
.1
0
-.1
-.2
-.3
-20000
-15000
-10000
-5000
0
Date
white noise
.04
.02
0
-.02
-.04
-20000
-15000
-10000
-5000
0
Date

ARCH(1)

- Unconditional variance of ε_t is constant

$$
\begin{aligned}
Var\left(\varepsilon_t\right) &= E\left(\varepsilon_t^2\right) = \varepsilon_t = E\left(u_t^2\right) E\left(\theta_0 + \theta_1 \varepsilon_{t-1}^2\right) \\
&= \theta_0 + \theta_1 Var\left(\varepsilon_{t-1}^2\right)
\end{aligned}
$$

- If the data generating process is stationary then $Var\left(\varepsilon_t^2\right) = Var\left(\varepsilon_{t-1}^2\right) = \ldots$
- Solving for $Var\left(\varepsilon_t\right)$ we obtain:

$$
Var\left(\varepsilon_t^2\right) = \frac{\theta_0}{1 - \theta_1}
$$

- Assumptions of CLRM are fullfiled - *OLS* is best linear unbiased estimator of β

ARCH cont.

- But conditional heteroscedasticity make possible to find more efficient *nonlinear* eestimators of β
- Variance of ε_t is not constant conditionally:

$$Var\left(\varepsilon_t \middle| \varepsilon_{t-1}\right) = E\left(\varepsilon_t^2 \middle| \varepsilon_{t-1}\right) = \theta_0 + \theta_1 \varepsilon_{t-1}^2$$

- General form of $ARCH(q)$

$$\begin{aligned} y_t &= \mathbf{x}_t \beta + \varepsilon_t \\ \varepsilon_t &= u_t \sigma_t \\ \sigma_t^2 &= \theta_0 + \theta_1 \varepsilon_{t-1}^2 + \ldots + \theta_q \varepsilon_{t-q}^2 \end{aligned}$$

- Notice that model discribin variance is similar to $MA(q)$

ARCH test

- Test for conditional heteroscedasticity of order q:

1. make regression of y_t on $\mathbf{x}_t$
2. squere residuals from this regression
3. make auxliary regression of e_t^2 on $e_{t-1}^2, \ldots, e_{t-q}^2$
4. calculate statitic $LM = TR^2 \sim \chi_q^2$

- $GARCH(p,q)$ Model (**G**eneralized **A**utoregressive **C**onditional **H**eteroskedasticity) is similar to $ARMA(p,q)$ model

$$\begin{aligned} y_t &= \mathbf{x}_t \beta + \varepsilon_t \\ \varepsilon_t &= u_t \sigma_t \\ \sigma_t^2 &= \alpha_1 \sigma_{t-1}^2 + \alpha_2 \sigma_{t-2}^2 + \ldots + \alpha_2 \sigma_{t-p}^2 + \theta_0 + \theta_1 \varepsilon_{t-1}^2 + \ldots + \theta_q \varepsilon_{t-q}^2 \end{aligned}$$

ARCHM

- *ARCHM* (*ARCH* in **M**eans) model:

$$\begin{aligned} y_t &= \mathbf{x}_t\beta + \delta\sigma_t^2 + \varepsilon_t \\ \varepsilon_t &= u_t\sigma_t \\ \sigma_t^2 &= \theta_0 + \theta_1\varepsilon_{t-1}^2 + \ldots + \theta_q\varepsilon_{t-q}^2 \end{aligned}$$

- In *ARCHM* model changes of variance influence mean
- This model is used in finance where it is often assumed that prices depend on risk (variance)
- Parameterδ is then related to risk aversion
- As higher risk should be compensated with higher expected return, then this parameter should be positive