Switchin regression and MSM models

Jerzy Mycielski

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Switching regression - model

Cross-section sample

- Regression equation

$$y_i = \mathbf{x}_{0i}\boldsymbol{\beta}_0 + u_{0i} \quad \text{for} \quad s_i = 0 \qquad \text{(State 0)}$$

$$y_i = \mathbf{x}_{1i}\boldsymbol{\beta}_1 + u_{1i} \quad \text{for} \quad s_i = 1 \qquad \text{(State 1)}$$

- Selection equation - model for binary variable

$$s_i = 0 \quad \text{for} \quad \mathbf{z}_i\gamma < u_i$$

$$s_i = 1 \quad \text{for} \quad \mathbf{z}_i\gamma \geq u_i$$

- Idnetifiaction: among explanatory variables in selection equation we should inlude variables, which are not included in regression equation

Swiching regression - random error assumptions

- We assume that random errors are *IID* - no autocorrelation (independence) and distributions are the same for all u_i
- Variance matrix of error terms

$$\Sigma = \begin{bmatrix} \sigma_{00} & \sigma_{02} & \sigma_{0u} \\ & \sigma_{11} & \sigma_{1u} \\ & & 1 \end{bmatrix}$$

- Usually we assume as well normality of error term

Switching regression - estimation

OLS estimation

- We assume normality of error term
- Fundamental problem, correlation between u_{i0}, u_{i1} and u_i

$$\mathrm{E}\left(u_{i0} \middle| \boldsymbol{z}_i\gamma \geq u_i \right) = -\sigma_{0u}\frac{\phi\left(\boldsymbol{z}_i\gamma\right)}{\Phi\left(\boldsymbol{z}_i\gamma\right)}$$

$$\mathrm{E}\left(u_{i1} \middle| \boldsymbol{z}_i\gamma < u_i \right) = \sigma_{1u}\frac{\phi\left(\boldsymbol{z}_i\gamma\right)}{\Phi\left(\boldsymbol{z}_i\gamma\right)}$$

- Conclusion: if $\boldsymbol{z}_i$ includes some of the variables included in $\boldsymbol{x}_{1i}$ or $\boldsymbol{x}_{2i}$ then the condition, $\mathrm{E}\left(u_{1i} \middle| \boldsymbol{x}_{1i} \right) = 0$ and $\mathrm{E}\left(u_{2i} \middle| \boldsymbol{x}_{2i} \right) = 0$ is invalid.

Switching regression - estimation

ML estimator

- Likelihood function:

$$L\left(\beta_0, \beta_1, \sigma_{00}, \sigma_{11}, \sigma_{01}, \sigma_{0u}, \sigma_{1u}\right)$$
$$= \prod_{i=1}^{N} \left[\int_{-\infty}^{\boldsymbol{z}_i \beta} f\left(y_i - \boldsymbol{x}_{1i} \boldsymbol{\beta}_1 \middle| u_i \right) du_i \right]^{s_i}$$
$$\times \left[\int_{\boldsymbol{z}_i \beta}^{\infty} f\left(y_i - \boldsymbol{x}_{2i} \boldsymbol{\beta}_1 \middle| u_i \right) du_i \right]^{s_i - 1}$$

Switching regression - special cases

- Sometimes *ML* estimation is difficult
- It is possible to use two-step estimator similar to the one used in Heckman model

Markov chain definition

- Values of random variable X_n are in finite set $\{s_1, s_2, \ldots, s_r\}$
- Finite Markov chain

$$\Pr\left(X_n = j \middle| X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}\right) = P\left(X_n = j \middle| X_{n-1} = i\right)$$

- This process has a short memory, transition probability only depends on the state of the process in previous period
- Finite homogenous Markov chain has a following property:

$$P\left(X_n = j \middle| X_{n-1} = i\right) = P\left(X_{n+s} = j \middle| X_{n+s-1} = i\right) = p_{ij}$$

- For homogenous Markov chain, transition probabilities do not change over time

- Matrix of transition probabilities has a following form:

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \vdots & \vdots & & \vdots \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{bmatrix}$$

- p_{ij} has to satisfy following properties:

$$\sum_{j=1}^{r} p_{ij} = 1$$

$$p_{ij} \geq 0$$

Markov switching model (MSM)

- Regression equation

$$
\begin{aligned}
y_t &= \boldsymbol{x}_t \boldsymbol{\beta}_0 + \sigma_0 \varepsilon_t \quad \text{for} \quad s_t = 0 && \text{(State 0)} \\
y_t &= \boldsymbol{x}_t \boldsymbol{\beta}_1 + \sigma_1 \varepsilon_t \quad \text{for} \quad s_t = 1 && \text{(State 1)}
\end{aligned}
$$

- Selection equation - Marcov chain with changing transition probabilities (nonhomogenous)

$$
P\left(s_t = i \middle| s_{t-1} = j, \boldsymbol{z}_t \right) = p_{ij}\left(\boldsymbol{z}_t \right)
$$

- We assume that

$$s_t = 0 \quad \text{for} \quad \boldsymbol{z}_t \gamma_{s_{t-1}} < u_t$$
$$s_t = 1 \quad \text{for} \quad \boldsymbol{z}_t \gamma_{s_{t-1}} \geq u_t$$

so that

$$p_{s_{t-1},0}\left(\boldsymbol{z}_t\right) = 1 - \Phi\left(\boldsymbol{z}_t \gamma_{s_{t-1}}\right)$$
$$p_{s_{t-1},1}\left(\boldsymbol{z}_t\right) = \Phi\left(\boldsymbol{z}_t \gamma_{s_{t-1}}\right)$$

Transition matrix

$$P\left(\boldsymbol{z}_t\right) = \begin{bmatrix} 1 - \Phi\left(\boldsymbol{z}_t \gamma_0\right) & \Phi\left(\boldsymbol{z}_t \gamma_0\right) \\ 1 - \Phi\left(\boldsymbol{z}_t \gamma_1\right) & \Phi\left(\boldsymbol{z}_t \gamma_1\right) \end{bmatrix}$$

- We assume that unconditional Markov chain is stationary:

$$P\left(s_t = i \middle| \overline{\mathbf{z}}\right) = P\left(s = i \middle| \overline{\mathbf{z}}\right) \text{ for every } t$$

$$\begin{bmatrix} \varepsilon_t \\ u_t \end{bmatrix} \sim N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

- We assume as well that the state in time t is unobservable

MSM - estimation

- Estimation usually done with *ML*
- In general case likelihood function complicated
- Special cases
  - exogenous switching: $\rho = 0$
  - pure Makov chain $\gamma_0 = \gamma_1 = \mathbf{0}$

- Joint density function for the sample

$$L(\boldsymbol{\theta}) = \prod_{t=1}^{T} f\left(y_t | \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta} \right)$$

where $\boldsymbol{\xi}_t = (y_t, y_{t-1}, \ldots, y_1)$,
$\boldsymbol{\Omega}_t = (\boldsymbol{x}_t, \boldsymbol{x}_{t-1}, \ldots, \boldsymbol{x}_1, \boldsymbol{z}_t, \boldsymbol{z}_{t-1}, \ldots, \boldsymbol{z}_1)$,
$\boldsymbol{\theta} = (\boldsymbol{\beta}_0, \sigma_0, \gamma_0, \boldsymbol{\beta}_1, \sigma_1, \gamma_1, \rho)$.

- Density function for one observation

$$f\left(y_t | \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta} \right)$$
$$= \sum_i \sum_j f\left(y_t | s_t = i, s_{t-1} = j, \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta} \right) \times$$
$$\Pr\left(s_t = i, s_{t-1} = j | \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta} \right)$$

- Transition probability for a given observation we calculate recursively from Bayes theorem and using that

$$P\left(s_0 = i \middle| \bar{z}\right) = P\left(s = i \middle| \bar{z}\right)$$

- so that:

$$\Pr\left(s_t = i, s_{t-1} = j \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right) = p_{ij}\left(z_t\right)\Pr\left(s_{t-1} = j \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right)$$

MSM - estimation

Transition probabilities cont.

- Unconditional transition probabilities we calculate from Bayes theorem as follows:

$$
\begin{aligned}
&\Pr\left(s_t = i \middle| \boldsymbol{\Omega}_{t+1}, \boldsymbol{\xi}_t; \boldsymbol{\theta}\right) \\
&= \Pr\left(s_t = i \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_t; \boldsymbol{\theta}\right) \\
&= \frac{f\left(y_t, s_t = i \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right)}{f\left(y_t \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right)} \\
&= \frac{1}{f\left(y_t \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right)} \times \\
&\sum_j f\left(y_t, s_t = i \middle| s_{t-1} = j, \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right) \times \\
&\Pr\left(s_t = i, s_{t-1} = j \middle| \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right)
\end{aligned}
$$

MSM - estimation

Transition probabilities cont.

- in case, when $\rho = 0$

$$f\left(y_t \middle| s_t = i, s_{t-1} = j, \boldsymbol{\Omega}_t, \boldsymbol{\xi}_{t-1}; \boldsymbol{\theta}\right) = \frac{1}{\sigma_{s_{t-1}}} \phi\left(\frac{y_t - \boldsymbol{x}_t \boldsymbol{\beta}_{s_{t-1}}}{\sigma_{s_{t-1}}}\right)$$

- In general case the formula is more complex