Stationary and nonstationary variables

- Stationary variable:
  1. Finite and constant in time expected value:

\[ \mathbb{E}(y_t) = \mu < \infty \]

  2. Finite and constant in time variance:

\[ \text{Var}(y_t) = \sigma^2 < \infty \]

  3. Covariance dependent only on \( h \) - distance in time

\[ \text{Cov}(y_t, y_{t+h}) = \gamma_h \]
for all $t$. Covariances depend only on $h$ distance in time between $y_t$ and $y_{t+h}$

- We call $y_t$ trend stationary if $y_t - E(y_t)$ is stationary

- Classical example of the stationary process is white noise:
  1. $y_t$ distribution is constant in time with expected value $0$ and finite variance $\sigma^2$
  2. $y_t$ and $y_s$ are independent
- white noise (variable $I(0)$)
Integrated variables

- Special case of stationary variables are integrated variables of order 0 $x_t \sim I(0)$. $I(0)$ variable is defined as stationary variable, which can be expressed as $x_t = \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$, where $\varepsilon_i$ and $\varepsilon_j$ are not correlated.

- Variable integrated of order $d$ denoted as $x_t \sim I(d)$, is a variable, that after applying $d$ order differences differences becomes integrated of order 0: $\Delta^d x_t \sim I(0)$.

- It can be shown that $A(L)^{-1} \varepsilon_t$ is stationary if all roots of $A(L)$ lie outside unit circle.

- If $x_t$ can be expressed as

$$A(L)x_t = \mu + \varepsilon_t$$
and $A(L)$ has all the root outside unit circle than $x_t$ is $I(0)$ as:

$$x_t = \mu^* + A(L)^{-1} \varepsilon_t$$

where $\mu^* = A(L)^{-1} \mu = A(1)^{-1} \mu = \frac{\mu}{1-\alpha_1-...-\alpha_p}$

- If $x_t$ is $I(1)$ then

  $$\Delta x_t = \mu^* + A(L)^{-1} \varepsilon_t$$

  or

  $$(1 - L) A(L) x_t = \mu + \varepsilon_t$$
  $$A^*(L) x_t = \mu + \varepsilon_t$$

  and $A^*(L)$ has one unit root and all other roots outside unit circle

- Conclusion: $I(1)$ variables has one unit root in their polynomial of lag operator.
Example 1. Random walk (nonstationy variable)

\[ y_t = y_{t-1} + \varepsilon_t \]

\[ \varepsilon_t \sim IID \left( 0, \sigma^2 \right) \]

This model can be written as

\[ (1 - L) y_t = \varepsilon_t \]

Obviously polynomial of lag operator has one unit root. \( A(L) \) is not invertible.

Substitute \( y_{t-1} = y_{t-2} + \varepsilon_{t-1} \),

\[ y_t = y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \]
Repeating this substitution recursively we will obtain:

$$y_t = y_0 + \sum_{s=1}^{t} \varepsilon_s$$

Assume that $y_0 = 0$, and so

$$\mathbb{E}(y_t) = 0$$

$$\text{Var}(y_t) = \sum_{s=1}^{t} \text{Var}(\varepsilon_s) = t\sigma^2$$

$$\text{Cov}(y_t, y_{t-h}) = \sum_{s=1}^{t-h} \text{Var}(\varepsilon_s) = (t-h)\sigma^2$$

Variances and covariances of the random walk depend on time! Variable $y_t$ is nonstationary.
Substracting $y_{t-1}$ from both sides we obtain:

$$y_t - y_{t-1} = \Delta y_t = \varepsilon_t$$

So that after taking first differences of random walk we obtain white noise - $I (0)$ variable. Conclusion: random walk is $I (1)$ variable.
• random walk ($I(1)$ variable)
• In econometric analysis only $I(1)$ and $I(2)$ are of importance

• A number of macroeconomic time series look like $I(1)$ variables

• In well known article Nelson and Plosser (1982) show, that significant part of macroeconomic time series for US look like $I(1)$ variables

• It is possible to test for order of integration

• As the order of integration is equal to number of unit roots this tests are called unit root tests
Dickey-Fuller Test ($DF$)

- We test the null hypothesis that the variable has one unit root (is $I(1)$)

- The simplest case

$$ y_t = \beta y_{t-1} + \varepsilon_t $$

- $H_0 : \beta = 1$, than $y_t$ has one unit root (is a random walk - $I(1)$ variable)

- $H_1 : |\beta| < 1$, then $y_t$ is stationary as the only root lies outside the unit circle

- Subtracting $y_{t-1}$ from both sides we obtain:

$$ \Delta y_t = (\beta - 1) y_{t-1} + \varepsilon_t $$

$$ = \rho y_{t-1} + \varepsilon_t $$
• $H_0: \rho = 0$, then $y_t$ is nonstationary

• $H_0: \rho \in (-2, 0)$, then $y_t$ is stationary

• Hypothesis than $\rho = 0$ can not be tested with standard $t$-student tables - the special test tables for $DF$ test should be used

• The test procedure:

  1. Regress $\Delta y_t$ on $y_{t-1}$
  2. Compare the $t$ statistic calculated for $y_{t-1}$ with critical values of $DF$ test. If calculated statistic is smaller than critical value we reject the $H_0$.

**Remark 2.** In Dickey Fuller the null and alternative hypotheses are as follows:

• $H_0$ variables has unit root (is nonstationary)
• $H_1$ variable has root outside unit circle (is stationary)

**Remark 3.** *In same tables of DF critical values minus sign is omitted.*
Augmented Dickey-Fuller test ($ADF$)

- Often in the regression $\Delta y_t = \rho y_{t-1} + \varepsilon_t$ there is autocorrelation

- Autocorrelation should be eliminated from dynamic model - otherwise we have simultaneity

- In this case we use $DF$ with augmentation ($ADF$ - Augmented Dickey Fuller)

  \[ \Delta y_t = \rho y_{t-1} + \sum_{i=1}^{k} \gamma_i \Delta y_{t-i} + \varepsilon_t \]

- $k$ is chosen so that autocorrelation is eliminated from residuals
• The constant and trend can also be included in the regression:

1. \[ \Delta y_t = \rho y_{t-1} + \sum_{i=1}^{k} \gamma_i \Delta y_{t-i} + \varepsilon_t \]
2. \[ \Delta y_t = a + \rho y_{t-1} + \sum_{i=1}^{k} \gamma_i \Delta y_{t-i} + \varepsilon_t \]
3. \[ \Delta y_t = a + bt + \rho y_{t-1} + \sum_{i=1}^{k} \gamma_i \Delta y_{t-i} + \varepsilon_t \]

Remark 4. For each trend model (no constant, constant, linear trend) we should use separate DF table

Example 5. Real GDP growth for Poland in years 1996.1-2004.3, quarterly index (Q/Q) - real growth quarter to the same quarter previous year (no seasonality)
• Growth of $PKB$ quarterly index (Q/Q)
| D.pkb | Coef.   | Std. Err. | t     | P>|t| | [95% Conf. Interval] |
|-------|---------|-----------|-------|-----|-------------------------|
| pkb   |         |           |       |     |                         |
| L1    | -.2760446 | .1350873  | -2.04 | 0.056 | -.5598524 , .0077633    |
| LD    | .5846543  | .2200515  | 2.66  | 0.016 | .1223433 , 1.046965     |
| L2D   | -.0616403 | .2488176  | -0.25 | 0.807 | -.5843866 , .461106     |
| L3D   | .3055767  | .2634729  | 1.16  | 0.261 | -.2479593 , .8591128    |
| L4D   | -.3489368 | .2367631  | -1.47 | 0.158 | -.8463575 , .148484     |
| L5D   | .2966282  | .2515727  | 1.18  | 0.254 | -.2319064 , .8251628    |
| L6D   | -.0437733 | .271078   | -0.16 | 0.874 | -.6132871 , .5257405    |
| L7D   | .1129747  | .2414823  | 0.47  | 0.646 | -.3943609 , .6203102    |
| _cons | .9466827  | .54444    | 1.74  | 0.099 | -.1971433 , 2.090509    |

Breusch-Godfrey LM test for autocorrelation

<table>
<thead>
<tr>
<th>lags(p)</th>
<th>chi2</th>
<th>df</th>
<th>Prob &gt; chi2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.568</td>
<td>1</td>
<td>0.4511</td>
</tr>
</tbody>
</table>

H0: no serial correlation
Augmented Dickey-Fuller test for unit root

Number of obs = 27

--- Interpolated Dickey-Fuller ---

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>1% Critical Value</th>
<th>5% Critical Value</th>
<th>10% Critical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z(t)</td>
<td>-2.043</td>
<td>-3.736</td>
<td>-2.994</td>
</tr>
</tbody>
</table>

* MacKinnon approximate p-value for Z(t) = 0.2679
Cointegrated variables

- Variables $x_{1t} \sim I(1)$ and $x_{2t} \sim I(1)$ are cointegrated if there is such $\beta$, that $x_{1t} + \beta x_{2t}$ is $I(0)$.

- In case of vector $x_t$, if each element of this vector is $I(1)$, then $x_t$ is cointegrated if there exists such vector $\beta$, that $\beta' x_t$ jest $I(0)$

- Cointegration between $I(1)$ variables exists if there is such linear combination of this variables which is $I(0)$
- cointegrated variables
Spurious regression

- Spurious regression can happen if some of the variables (explanatory or dependent) in the model are not $I(0)$

- In such a regression we can obtain significant $t$ statistics in $OLS$ regression even for variables which are totally unrelated

- Regress $y_t$ on $x_t$, where $y_t$ and $x_t$ are independent $I(1)$ variables (random walks).

  \[
y_t = \beta_0 + \beta_1 x_t + e_t
  \]

- $p$-values taken from $t$-student distribution and simulated $p$-values for the
test for $H_0 : \beta_1 = 0$.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{obs. number} & t_\alpha & \text{t-student} & \text{simulation} \\
\hline
10 & 2.31 & 0.05 & 0.28 \\
100 & 1.98 & 0.05 & 0.76 \\
1000 & 1.96 & 0.05 & 0.93 \\
\hline
\end{array}
\]

- For large number of observation it is virtually sure that $|t| > t_\alpha$ and we reject true null hypothesis.

- Using standard regression procedures for models with nostationary variables we can easily end up with a model with irrelevant variables (so the term spurious regression)

**Remark 6.** Although the $t$ statistic has nonstandard distribution in this case, the OLS estimators are still consistent
• Simple solution of the spurious regression problem for the regression with I(1) variables: take the first differences of both sides (make the regression on first differences)

- Problem: regression on first deferences can not be used for estimating the long run equilibrium:

\[
\Delta y_t = \Delta x_t \beta + \varepsilon_t \\
E(y_t - y_{t-1}) = E(x_t - x_{t-1}) \beta
\]

and assuming that \(y^* = E(y_t) = E(y_{t-1}) = \ldots\) and \(x^* = E(x_t) = E(x_{t-1}) = \ldots\) we obtain equation \(0 = 0\beta\) which can not be used to find the long run solution (long rung solution for levels cannot be calculated from model on first differences).