Stationary and nonstationary variables

- Stationary variable:

1. Finite and constant in time expected value:

$$\mathrm{E}\left(y_t\right) = \boldsymbol{\mu} < \infty$$

2. Finite and constant in time variance:

$$\mathrm{Var}\left(y_t\right) = \boldsymbol{\sigma}^2 < \infty$$

3. Covariance dependent only on h - distance in time

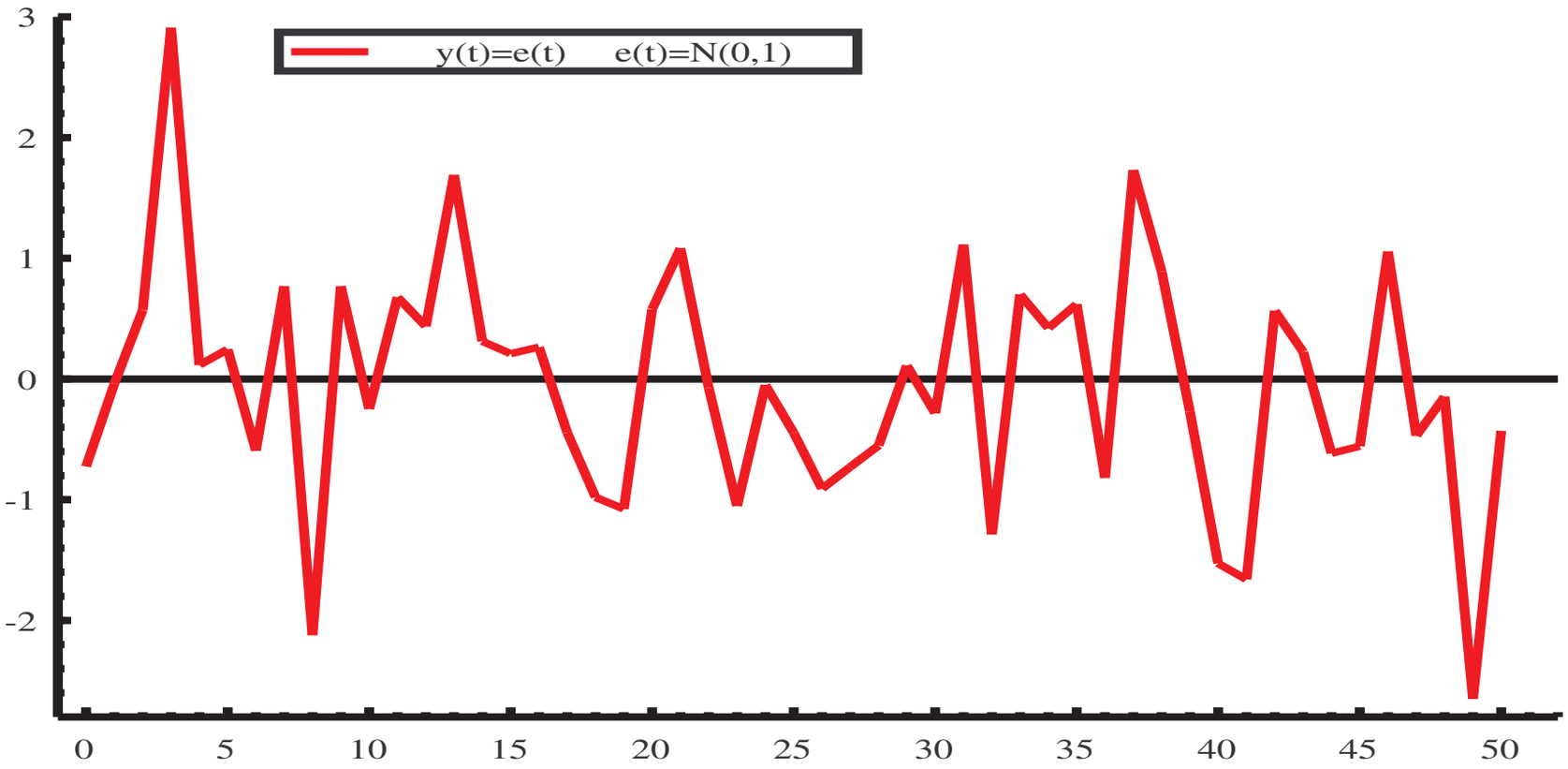
$$\mathrm{Cov}\left(y_t, y_{t+h}\right) = \boldsymbol{\gamma}_h$$

for all t. Covariances depend only on h distance in time between y_t and y_{t+h}

- We call y_t trend stationary if $y_t - \mathrm{E}(y_t)$ is stationary

- Classical example of the stationary process is white noise:

  1. y_t distribution is constant in time with expected value 0 and finite variance $\boldsymbol{\sigma}^2$
  2. y_t and y_s are independent



- white noise (variable $I(0)$)

Integrated variables

- Special case of stationary variables are integrated variables of order 0 $x_t \sim I(0)$. $I(0)$ variable is defined as stationary variable, which can be expressed as $x_t = \sum_{i=0}^{\infty} c_i \boldsymbol{\varepsilon}_{t-i}$, where $\boldsymbol{\varepsilon}_i$ and $\boldsymbol{\varepsilon}_j$ are not correlated.

- Variable integrated of order d denoted as $x_t \sim I(d)$, is a variable, that after applying d order differences differences becomes integrated of order 0: $\Delta^d \boldsymbol{x}_t \sim I(0)$.

- It can be shown that $A(L)^{-1} \boldsymbol{\varepsilon}_t$ is stationary if all roots of $A(L)$ lie outside unit circle.

- If x_t can be expressed as

$$A(L)\, x_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t$$

and $A(L)$ has all the root outside unit circle than x_t is $I(0)$ as:

$$x_t = \boldsymbol{\mu}^* + A(L)^{-1} \boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\mu}^* = A(L)^{-1} \boldsymbol{\mu} = A(1)^{-1} \boldsymbol{\mu} = \frac{\boldsymbol{\mu}}{1-\boldsymbol{\alpha}_1-\ldots-\boldsymbol{\alpha}_p}$

- If x_t is $I(1)$ then

$$\Delta x_t = \boldsymbol{\mu}^* + A(L)^{-1} \boldsymbol{\varepsilon}_t$$

or

$$\begin{aligned} (1-L) A(L) x_t &= \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \\ A^*(L) x_t &= \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \end{aligned}$$

and $A^*(L)$ has one unit root and all other roots outside unit circle

- Conclusion: $I(1)$ variables has one unit root in their polynomial of lag operator.

Example 1. *Random walk (nonstationy variable)*

$$y_t = y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim IID\left(0, \boldsymbol{\sigma}^2\right)$$

This model can be written as

$$(1 - L)\, y_t = \varepsilon_t$$

Obviously polynomial of lag operator has one unit root. $A(L)$ *is not invertible.*

Substitute $y_{t-1} = y_{t-2} + \varepsilon_{t-1},$

$$y_t = y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

Repeating this substitution recursively we will obtain:

$$y_t = y_0 + \sum_{s=1}^{t} \boldsymbol{\varepsilon}_s$$

Assume that $y_0 = 0$*, and so*

$$\mathrm{E}\left(y_t\right) = 0$$

$$\mathrm{Var}\left(y_t\right) = \sum_{s=1}^{t} \mathrm{Var}\left(\boldsymbol{\varepsilon}_s\right) = t\boldsymbol{\sigma}^2$$

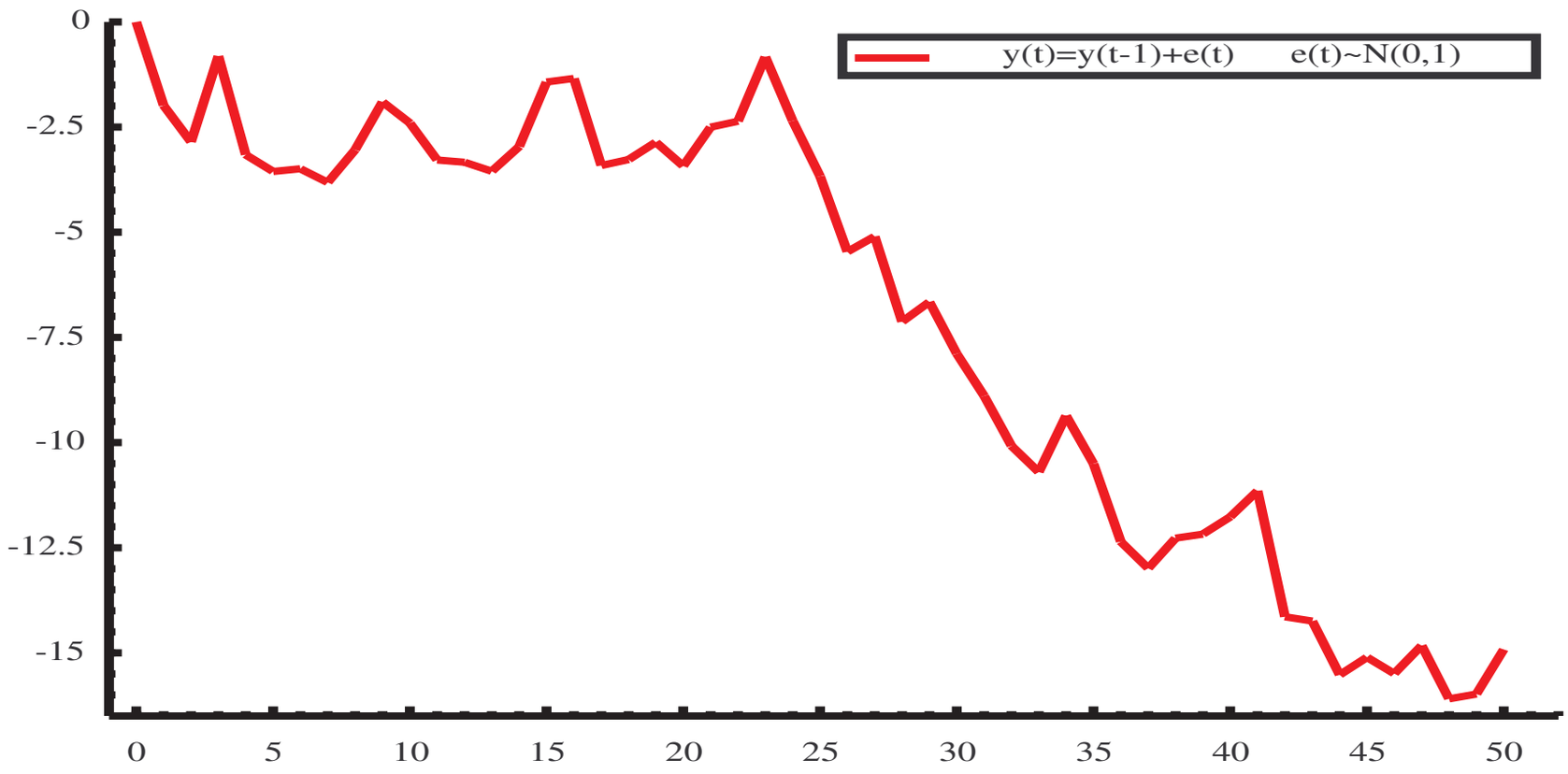
$$\mathrm{Cov}\left(y_t, y_{t-h}\right) = \sum_{s=1}^{t-h} \mathrm{Var}\left(\boldsymbol{\varepsilon}_s\right) = \left(t-h\right)\boldsymbol{\sigma}^2$$

Variances and covariances of the random walk depend on time! Variable y_t *is nonstationary.*

Substracting y_{t-1} *from both sides we obtain:*

$$y_t - y_{t-1} = \Delta y_t = \varepsilon_t$$

So that after taking first differences of random walk we obtain white noise - $I(0)$ *variable. Conclusion: random walk is* $I(1)$ *variable.*



- random walk ($I(1)$ variable)

- In econometric analysis only $I(1)$ and $I(2)$ are of importance
- A number of macroeconomic time series look like $I(1)$ variables
- In well known article Nelson and Plosser (1982) show, that significant part of macroeconomic time series for US look like $I(1)$ variables
- It is possible to test for order of integration
- As the order of integration is equal to number of unit roots this tests are called unit root tests

Dickey-Fuller Test (DF)

- We test the null hypothesis that the variable has one unit root (is $I(1)$)
- The simplest case

$$y_t = \boldsymbol{\beta} y_{t-1} + \boldsymbol{\varepsilon}_t$$

- $H_0 : \boldsymbol{\beta} = 1$, than y_t has one unit root (is a random walk - $I(1)$ variable)
- $H_1 : |\boldsymbol{\beta}| < 1$, then y_t is stationary as the only root lies outside the unit circle
- Substracting y_{t-1} from both sides we obtain:

$$\begin{aligned} \Delta y_t &= (\boldsymbol{\beta} - 1) y_{t-1} + \boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\rho} y_{t-1} + \boldsymbol{\varepsilon}_t \end{aligned}$$

- $H_0 : \boldsymbol{\rho} = 0$, then y_t is nonstationary

- $H_0 : \boldsymbol{\rho} \in (-2, 0)$, then y_t is stationary

- Hypothesis than $\boldsymbol{\rho} = 0$ can not be tested with standard t-student tables - the special test tables for DF test should be used

- The test procedure:

  1. Regress Δy_t on y_{t-1}
  2. Compare the t statistic calculated for y_{t-1} with critical values of DF test. If calculated statistic is smaller than critical value we reject the H_0.

Remark 2. *In Dickey Fuller the null and alternative hypotheses are as follows:*

- H_0 *variables has unit root (is nonstationary)*

- H_1 *variable has root outside unit circle (is stationary)*

Remark 3. *In same tables of DF critical values minus sign is omitted.*

Augmented Dickey-Fuller test (ADF)

- Often in the regression $\Delta y_t = \boldsymbol{\rho} y_{t-1} + \boldsymbol{\varepsilon}_t$ there is autocorrelation
- Autocorrelation should be eliminated from dynamic model - otherwise we have simultaneity
- In this case we use DF with augmentation (ADF - Augmented **D**ickey **F**uller)

$$\Delta y_t = \boldsymbol{\rho} y_{t-1} + \sum_{i=1}^{k} \boldsymbol{\gamma}_i \Delta y_{t-i} + \boldsymbol{\varepsilon}_t$$

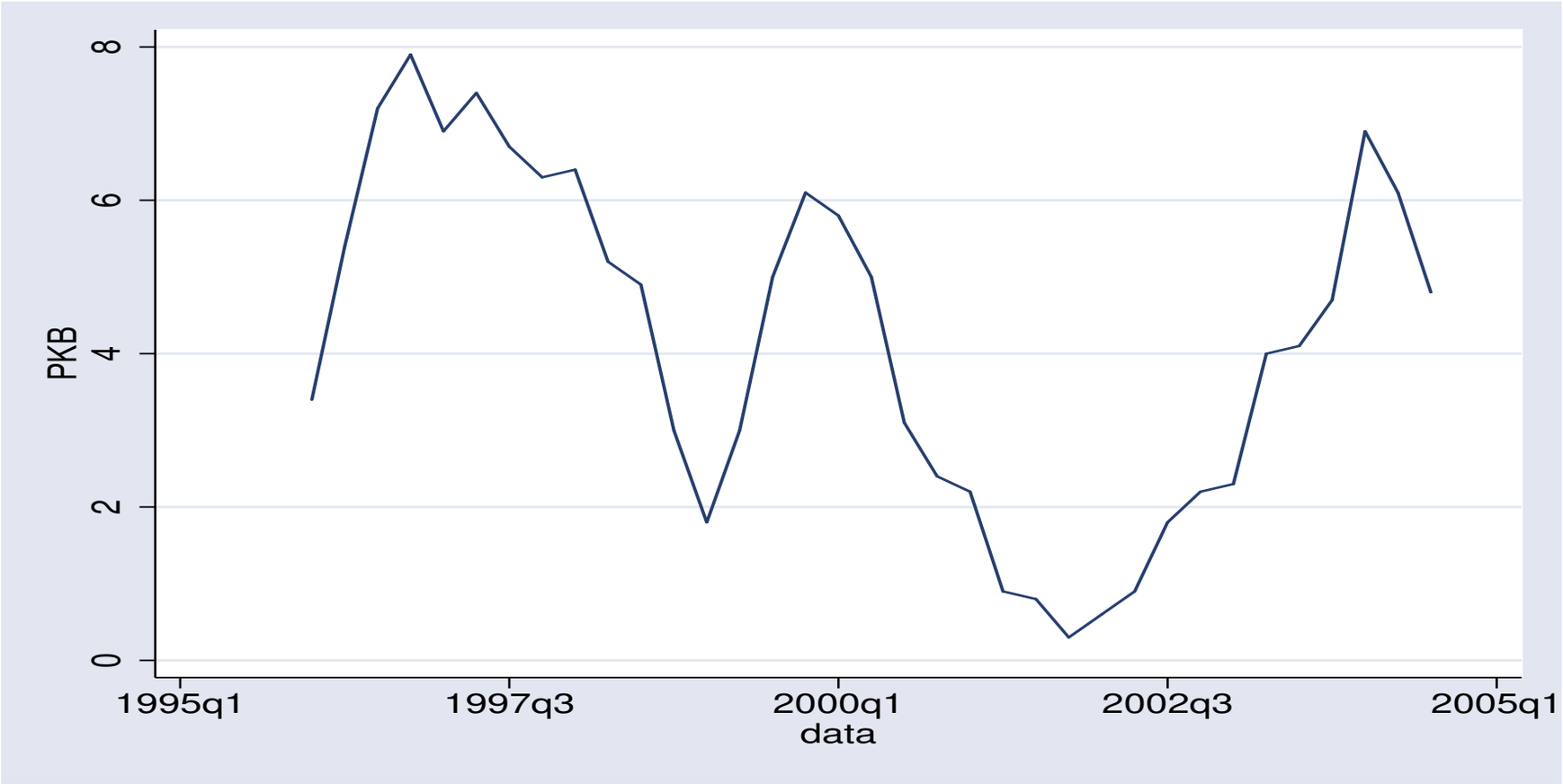
- k is chosen so that autocorrelation is eliminated from residuals

- The constant and trend can also be included in the regression:

1. $\Delta y_t = \boldsymbol{\rho} y_{t-1} + \sum_{i=1}^{k} \boldsymbol{\gamma}_i \Delta y_{t-i} + \boldsymbol{\varepsilon}_t$
2. $\Delta y_t = a + \boldsymbol{\rho} y_{t-1} + \sum_{i=1}^{k} \boldsymbol{\gamma}_i \Delta y_{t-i} + \boldsymbol{\varepsilon}_t$
3. $\Delta y_t = a + bt + \boldsymbol{\rho} y_{t-1} + \sum_{i=1}^{k} \boldsymbol{\gamma}_i \Delta y_{t-i} + \boldsymbol{\varepsilon}_t$

Remark 4. *For each trend model (no constant, constant, linear trend) we should use separate DF table*

Example 5. *Real GDP growth for Poland in years 1996.1-2004.3, quarterly index (Q/Q) - real growth quarter to the some quarter previous year (no seasonality)*



- Growth of PKB quarterly index (Q/Q)

D.pkb	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
pkb						
L1	-.2760446	.1350873	-2.04	0.056	-.5598524	.0077633
LD	.5846543	.2200515	2.66	0.016	.1223433	1.046965
L2D	-.0616403	.2488176	-0.25	0.807	-.5843866	.461106
L3D	.3055767	.2634729	1.16	0.261	-.2479593	.8591128
L4D	-.3489368	.2367631	-1.47	0.158	-.8463575	.148484
L5D	.2966282	.2515727	1.18	0.254	-.2319064	.8251628
L6D	-.0437733	.271078	-0.16	0.874	-.6132871	.5257405
L7D	.1129747	.2414823	0.47	0.646	-.3943609	.6203102
_cons	.9466827	.54444	1.74	0.099	-.1971433	2.090509

Breusch-Godfrey LM test for autocorrelation

lags(p)	chi2	df	Prob > chi2
1	0.568	1	0.4511

H0: no serial correlation

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Augmented Dickey-Fuller test for unit root         Number of obs   =        27

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -2.043            -3.736            -2.994            -2.628
------------------------------------------------------------------------------
* MacKinnon approximate p-value for Z(t) = 0.2679
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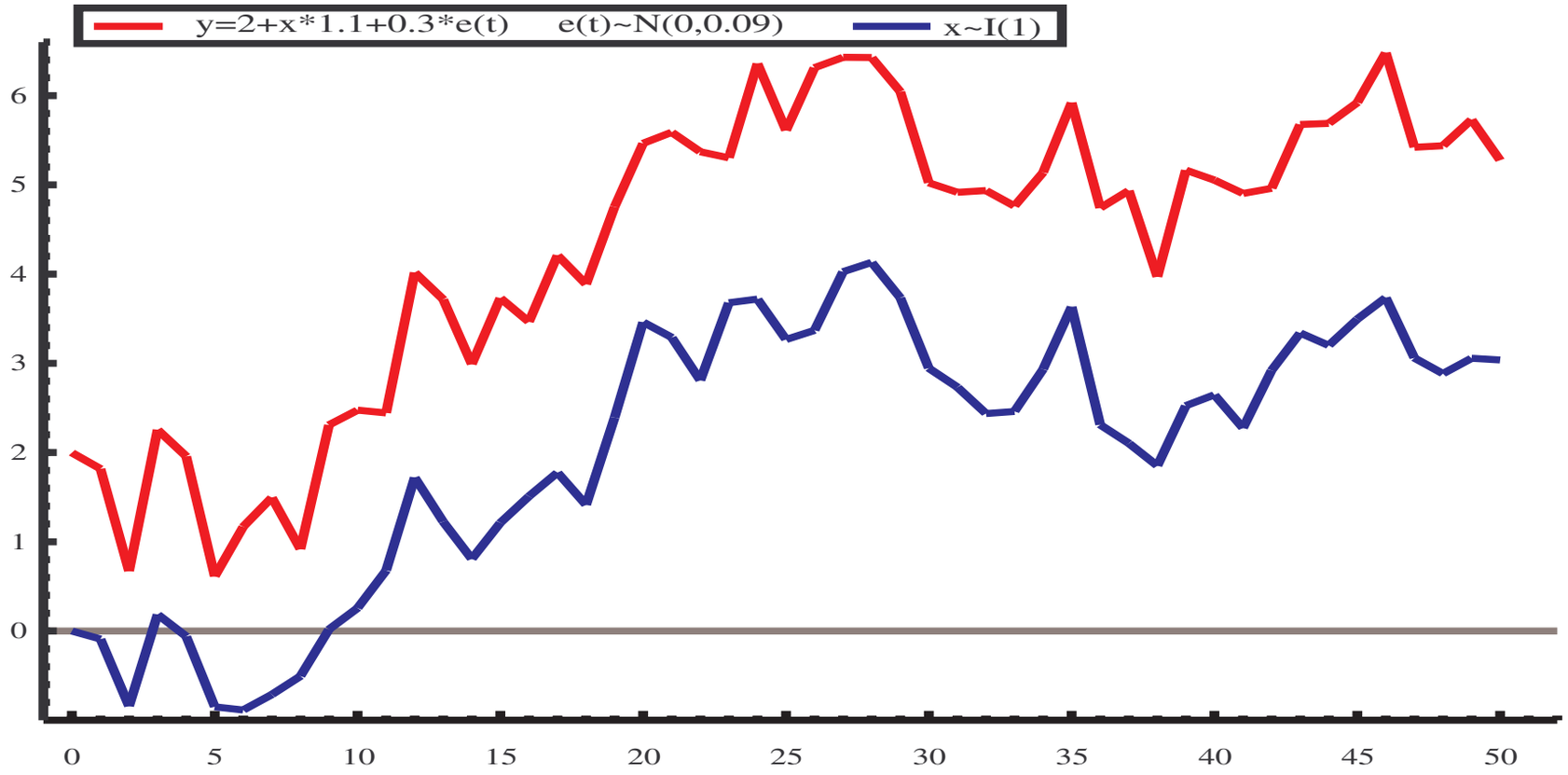
Cointegrated variables

- Variables $x_{1t} \sim I(1)$ and $x_{2t} \sim I(1)$ are cointegrated if there is such $\boldsymbol{\beta}$, that $x_{1t} + \boldsymbol{\beta} x_{2t}$ is $I(0)$.

- In case of vector $\boldsymbol{x}_t$, if each element of this vector is $I(1)$, then $\boldsymbol{x}_t$ is cointegrated if there exists such vector $\boldsymbol{\beta}$, that $\boldsymbol{\beta}' \boldsymbol{x}_t$ jest $I(0)$

- Cointegration between $I(1)$ variables exists if there is such linear combination of this variables which is $I(0)$



- cointegrated variables

Spurious regression

- Spurious regression can happen if some of the variables (explanatory or dependent) in the model are not $I(0)$

- In such a regression we can obtain significant t statistics in OLS regression even for variables which are totally unrelated

- Regress y_t on x_t, where y_t and x_t are *independent* $I(1)$ variables (random walks).

$$y_t = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 x_t + e_t$$

- $p-$values taken from t-student distribution and simulated p-values for the

test for $H_0 : \boldsymbol{\beta}_1 = 0$.

		probability, that $\lvert t \rvert > t_{\boldsymbol{\alpha}}$	
obs. number	$t_{\boldsymbol{\alpha}}$	t-student	simulation
10	2.31	0.05	0.28
100	1.98	0.05	0.76
1000	1.96	0.05	0.93

- For large number of observation it is virtually sure that $\lvert t \rvert > t_{\boldsymbol{\alpha}}$ and we reject *true* null hypothesis.

- Using standard regression procedures for models with nostationary variables we can easily end up with a model with irrelevant variables (so the term spurious regression)

Remark 6. *Although the t statistic has nonstandard distribution in this case, the OLS estimators are still consistent*

- Simple solution of the spurious regression problem for the regression with $I(1)$ variables: take the first differences of both sides (make the regression on first differences)
  - Problem: regression on first deferences can not be used for estimating the long run equilibrium:

$$\begin{aligned} \Delta y_t &= \Delta \boldsymbol{x}_t \boldsymbol{\beta} + \boldsymbol{\varepsilon}_t \\ \mathrm{E}\left(y_t - y_{t-1}\right) &= \mathrm{E}\left(\boldsymbol{x}_t - \boldsymbol{x}_{t-1}\right)\boldsymbol{\beta} \end{aligned}$$

    and assuming that $y^* = \mathrm{E}(y_t) = \mathrm{E}(y_{t-1}) = \ldots$ and $\boldsymbol{x}^* = \mathrm{E}(\boldsymbol{x}_t) = \mathrm{E}(\boldsymbol{x}_{t-1}) = \ldots$ we obtain equation $\mathbf{0} = \mathbf{0}\boldsymbol{\beta}$ which can not be used to find the long run solution (long rung solution for levels cannot be calculated from model on first differences).