Algebra revision - basic concepts

1. Matrix algebra: adding, multiplying, transposition
2. Properties of matrices' determinants
3. Quadratic forms, positive definite matrices
4. Trace and its properties
5. Idempotent matrix, its characteristic roots and its trace

Problems

1. We have a matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$
Find $2\mathbf{A}$, $\mathbf{B}'$, $\mathbf{A} + \mathbf{B}'$, $\mathbf{AB}$, $|\mathbf{AB}|$. Explain why we cannot find $\mathbf{AB}'$ i $\mathbf{A} + \mathbf{B}$.
2. We have two quadratic matrices
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$
Show that matrix $\mathbf{A}$ is symmetric. Show that $\mathbf{AB} \neq \mathbf{BA}$. Show that matrices $\mathbf{A}$ and $\mathbf{AB}$ are peculiar.
3. Extend (assume that $\mathbf{A}$ are $\mathbf{B}$ are invertible): $(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D})'$, $(\mathbf{AB})^{-1}\mathbf{B}^{-1}$, $(\mathbf{BA})^{-1}\mathbf{B}^{-1}$, $\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}$, $|\mathbf{AB}|$.
4. Show that for any invertible $\mathbf{A}$, $\left(\mathbf{A}^{-1}\right)' = (\mathbf{A}')^{-1}$.
5. Show (from the definition), that $\mathbf{X}'\mathbf{X}$ is nonnegative definite.
6. Show (applying linear independence) that $\mathbf{X}'\mathbf{X}$ is nonpeculiar if the columns of matrix $\mathbf{X}$ are linearly independent.
7. Prove that for matrix $\mathbf{A}$ i $\mathbf{B}$ with right dimensions $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.
8. Prove that the following holds for a trace of a matrix $tr(\mathbf{A} + \mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$, $tr(\mathbf{AB}) = tr(\mathbf{BA})$.
9. Show that for any idempotent $\mathbf{P}$, $\mathbf{M} = \mathbf{I} - \mathbf{P}$ is also idempotent. In addition, show that $\mathbf{MA} = \mathbf{0}$.
10. Show that for matrix $\mathbf{P}$ is idempotent for any $\mathbf{A}$ such that $\mathbf{A}'\mathbf{A}$ is nonpeculiar:
$$\mathbf{P} = \mathbf{A}\left(\mathbf{A}'\mathbf{A}\right)^{-1}\mathbf{A}'$$
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