Advanced Econometrics

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Normal regression model

- The normal regression model is the linear regression with an independent, homoskedastic normal error

$$\begin{aligned} Y &= X'\beta + e \\ e &\sim N(0,\sigma^2) \end{aligned}$$

- Likelihood is the joint (conditional) probability of the observed sample. It is denoted as $L()$.
- For the estimation of the model with fully specified distribution we usually use the ML estimator which maximises likelihood (or equivalently $lnL\left(\right) = \ell\left(\right)$)
- In the case of normal regression the loglikelihood is

$$\ell\left(\beta,\sigma^2\right) = lnL\left(\beta,\sigma^2\right) = -\frac{n}{2} log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y - X'\beta\right)^2$$

- The maximisation problem

$$\left(\widehat{\beta}_{mle}, \widehat{\sigma}^2_{mle}\right) = \underset{\beta\in\mathbb{R}^k,\sigma^2>0}{\operatorname{argmax}}\ lnL\left(\beta,\sigma^2\right)$$

Normal regression model

- Solving first order conditions we obtain

$$\widehat{\beta}_{mle} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y} = \widehat{\beta}_{ols}$$

$$\widehat{\sigma}^2_{mle} = \frac{1}{n}\sum_{i=1}^{n}\widehat{e}_i^2 = \widehat{\sigma}^2_{ols}$$

- Method of Moments estimators and ML estimators identical! This result is exceptional - in many cases these estimators are different.
- Under assumptions of normal regression model we can prove that

$$\widehat{\beta}\Big|\mathbf{X} \sim N\left(\beta, \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\right)$$

- Using normality of $\widehat{\beta}$ we prove that statistic T has t-student distribution

$$T\left(\beta\right) = \frac{\widehat{\beta}_j - \beta_j}{\sqrt{s^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}_{jj}}} = \frac{\widehat{\beta}_j - \beta_j}{se\left(\widehat{\beta}_j\right)} \sim t_{n-k}$$

Confidence intervals and t test

- Confidence interval

$$\widehat{C} = \left[\widehat{\beta} - c \times s(\widehat{\beta}), \widehat{\beta} + c \times s(\widehat{\beta})\right]$$

- Probability α of $\widehat{C}$ covering the true value of β is

$$\mathbb{P}\left[\beta \in \widehat{C}\right] = \mathbb{P}\left[-c < T(\beta) < c\right] = 2F(c) - 1 = 1 - \alpha$$

where $F()$ is cdf of $T(\beta)$ and α is known as confidence level

Significance test (t - test)

- Consider null H_0 and alternative hypotheses H_1

$$H_0 : \beta_j = \beta$$

$$H_1 : \beta_j \neq \beta$$

- We reject H_0 (and accept H_1) if $|T| > c$ as large value of $|T|$ suggests that $\widehat{\beta}$ deviates *significanly* from β.
- We can make two errors
  - reject H_0 which is true (type 1 error)
  - not reject H_0 which is false (type 2 error)
- Probability of type 1 error is equal to

$$\mathbb{P}\left[|T| > c|\, H_0 \text{ true}\right] = \alpha$$

  where α is called significance level
- Probability of type 2 error

$$\mathbb{P}\left[|T| < c|\, H_0 \text{ false}\right] = \beta$$

  $1 - \beta$ (probability of rejecting false H_0) is known as the power of the test

Bibliography