Simultaneous equation model (structural form)

- Structural form is such a form of the model in which all the equations and the parameters have ”deep” economic interpretation.
- Usually it means that the signs and sometimes the relations between the parameters can be established on the basis of the theory.
- The structural form can be used to verify the consistency of the theoretical model with the data.
- Structural form with G equations we write as

$$\boldsymbol{A}\boldsymbol{Y}_t = \boldsymbol{B}\boldsymbol{X}_t + \boldsymbol{u}_t,$$

where endogenous variables are denoted as vector

$$\boldsymbol{Y}_t = [y_{1t}, \ldots, y_{Gt}]'$$

exogenous variables as vector

$$\boldsymbol{X}_t = [x_{1t}, \ldots, x_{Kt}]'$$

error term is given by

$$\boldsymbol{u}_t = [u_{1t}, \ldots, u_{Gt}]'$$

- Typically we assume that:
  - expected values of the error terms are equal to zero

$$\mathrm{E}\left(\boldsymbol{u}_t\right) = \mathbf{0}$$

- simultaneous error terms can be correlated but the correlation structure is constant over time

$$\mathrm{Var}\left(\boldsymbol{u}_t\right) = \boldsymbol{\Sigma}$$

- there is no autocorrelation (correlation over time)

$$\mathrm{E}\left(\boldsymbol{u}_t \boldsymbol{u}_s\right) = \mathbf{0} \text{ for } t \neq s$$

- $\boldsymbol{Y}_t$ vector of endogenous variables
- $\boldsymbol{X}_t$ is vector of exogenous and predetermined variables
- We assume that matrix $\boldsymbol{A}_{G\times G}$ has full rank
- In one equation we can have more than one endogenous variable

Example 1. *Example (demand and supply - model on logarithms)*

$$\begin{array}{ll} Q_D = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 P + \boldsymbol{\alpha}_2 Y + u_1 & \textit{demand curve} \\ Q_S = \boldsymbol{\beta}_1 P + \boldsymbol{\beta}_2 P_M + u_2 & \textit{supply curve} \\ Q_D = Q_S & \textit{market clearing condition} \end{array}$$

Q_D-demand, Q_S-supply, P-price, Y consumers income, P_M-prices of raw materials. Interpretation of parameters: $\boldsymbol{\alpha}_1$ - price elasticity of demand, $\boldsymbol{\alpha}_2$ - income elasticity of demand, $\boldsymbol{\beta}_1$ - price elasticity of supply, $\boldsymbol{\beta}_2$ - raw material price elasticity of supply. The signs of the coefficients can be derived from theory ($\boldsymbol{\alpha}_1 < 0$, $\boldsymbol{\alpha}_2 > 0$ if good is a normal good, $\boldsymbol{\beta}_1 > 0$, $\boldsymbol{\beta}_2 < 0$)

What is endogenous and what is exogenous in this system of equations?

According to theory the market sets the price on such a level that the demand and supply at this price are equal. Hence Q_D, Q_S, P are *endogenous. This system written as* $\boldsymbol{A}\boldsymbol{Y}_t = \boldsymbol{B}\boldsymbol{X}_t + \boldsymbol{u}_t$ *has the form*

$$
\begin{aligned}
Q_D - \boldsymbol{\alpha}_1 P &= \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_2 Y + u_1 \\
Q_S - \boldsymbol{\beta}_1 P &= \boldsymbol{\beta}_2 P_M + u_2 \\
Q_D - Q_S &= 0
\end{aligned}
$$

For matrices $\boldsymbol{Y}_t = [Q_D, Q_S, P]'$, $\boldsymbol{X}_t = [1, Y, P_M]$, $\boldsymbol{u}_t = [u_{1t}, u_{2t}, 0]'$

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 1 & 0 & -\boldsymbol{\alpha}_1 \\ 0 & 1 & -\boldsymbol{\beta}_1 \\ 1 & -1 & 0 \end{bmatrix} \\
\boldsymbol{B} &= \begin{bmatrix} \boldsymbol{\alpha}_0 & \boldsymbol{\alpha}_2 & 0 \\ 0 & 0 & \boldsymbol{\beta}_2 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Simultaneous equation model (reduced form)

- Reduced form of the model is such form that:
  - in all equations at the left hand side we have one endogenous variable
  - on the right hand side only the exogenous variables.
- Reduced form of the structural model can be obtained from the structural form by simple algebraic transformation:

$$\boldsymbol{Y}_t = \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{X}_t + \boldsymbol{A}^{-1}\boldsymbol{u}_t$$

- Define matrix $\mathbf{\Pi} = \boldsymbol{A}^{-1}\boldsymbol{B}$ and error term $\boldsymbol{\varepsilon}_t = \boldsymbol{A}^{-1}\boldsymbol{u}_t$ and then

$$\boldsymbol{Y}_t = \mathbf{\Pi}\boldsymbol{X}_t + \boldsymbol{\varepsilon}_t$$

- The parameters in reduced form:

  - has no economic interpretation
  - can be interpreted as multipliers: the total reaction of the endogenous variable on the change of the dependent variable irrespective of the economic mechanism behind.
  - are often interesting in themselves as they can be used for policy analysis.

Example 2. *(demand and supply)*

$$Q_D = \boldsymbol{\pi}_{11} + \boldsymbol{\pi}_{12} Y + \boldsymbol{\pi}_{13} P_M + \boldsymbol{\varepsilon}_1 = Q_S$$
$$P = \boldsymbol{\pi}_{21} + \boldsymbol{\pi}_{22} Y + \boldsymbol{\pi}_{23} P_M + \boldsymbol{\varepsilon}_2$$

where $\boldsymbol{\pi}$ are following functions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$\boldsymbol{\pi}_{11} = \frac{\boldsymbol{\beta}_1 \boldsymbol{\alpha}_0}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1},\ \boldsymbol{\pi}_{12} = \frac{\boldsymbol{\beta}_1 \boldsymbol{\alpha}_2}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1},\ \boldsymbol{\pi}_{13} = -\frac{\boldsymbol{\beta}_2 \boldsymbol{\alpha}_1}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1}$$
$$\boldsymbol{\pi}_{21} = \frac{\boldsymbol{\alpha}_0}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1},\ \boldsymbol{\pi}_{22} = \frac{\boldsymbol{\alpha}_2}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1},\ \boldsymbol{\pi}_{23} = -\frac{\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1}$$

Interpretation of parameters: $\boldsymbol{\pi}_{12}$ - consumer income multiplier of the market turnover: reaction of the market turnover (demand/supply) on the change of the consumer incomes (it is obviously not the same as income elasticity of demand) Similarly: $\boldsymbol{\pi}_{22}$ consumer income multiplier of the market price.

The ε are linear combinations of u

$$
\begin{aligned}
\boldsymbol{\varepsilon}_1 &= \frac{\boldsymbol{\beta}_1}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1} u_1 - \frac{\boldsymbol{\alpha}_1}{\boldsymbol{\beta} -_{1} \boldsymbol{\alpha}_1} u_2 \\
\boldsymbol{\varepsilon}_2 &= \frac{1}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1} u_1 - \frac{1}{\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1} u_2
\end{aligned}
$$

Identification

- Multiply structural model from the left hand side by nonsingular matrix $\boldsymbol{F}_{G\times G}$
$$\boldsymbol{FAY}_t = \boldsymbol{FBX}_t + \boldsymbol{Fu}_t$$
- It can be written as
$$\boldsymbol{A}^*\boldsymbol{Y}_t = \boldsymbol{B}^*\boldsymbol{X}_t + \boldsymbol{u}_t^*$$
where $\boldsymbol{A}^* = \boldsymbol{FA}$, $\boldsymbol{B}^* = \boldsymbol{FB}$, $\boldsymbol{u}_t^* = \boldsymbol{Fu}_t$
- The model obtained in such a way is algebraically equivalent to the original model!
- No sample of observations can help to distinguish the transformed model from the original one.

- The parameters of the structural model can be estimated only if parameters of the model are identified

- For identification it is necessary to impose such restrictions on matrices $\boldsymbol{A}$, $\boldsymbol{B}$ that the matrices $\boldsymbol{A}^*$, $\boldsymbol{B}^*$ of the transformed model satisfy them only for $\boldsymbol{F} = \boldsymbol{I}$

- Model is called
  - not identified: the number of restrictions is too small to identify it
  - exactly identified: the number of restrictions is equal to the minimum number of restrictions needed to identify the model
  - overidentified: the number of restrictions is bigger then needed for identification

- Restrictions needed to identify the model can not be verified on the basis of data and then should be derived from theory

- Validity of the restrictions over the number needed for identification can be verified on the basis of the empirical data.
- Model which is not identified cannot be estimated
- Necessary condition for identification. For all the equations of the model the following condition is satisfied:

$$K \geq G_j + K_j - 1$$

  where K the total number of exogenous variables, G_j number of endogenous variables in j equation, K_j number of exogenous variables in j equation.
- This formula enable us to determine the identification of *equations*:
  - $K < G_j + K_j - 1$ equation is not identified

- $K = G_j + K_j - 1$ equation is exactly identified
- $K > G_j + K_j - 1$ equation is overidentified

- Even if the model is not identified but one of the equations is identified it is usually possible to estimate this equation.

Example 3. *(demand and supply). Consider the textbook supply/demand model. Is it possible to estimate it?*

$$Q_D = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 P + \boldsymbol{\varepsilon}_1$$
$$Q_S = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 P + \boldsymbol{\varepsilon}_2$$
$$Q_D = Q_S$$

It is infeasible to estimate this model as it is not identified. Demand and supply equations are observationally equivalent (dependent variable the *same, independent variable also the same). It is then impossible to determine whether we estimated the demand or supply equation.*

Check the necessary conditions for identification: first equation $1 < 2 + 1 - 1$*, second equation* $1 < 2 + 1 - 1$

But: it is possible to identify the demand and supply equations if in each of them are variables absent from the other one. For instance in our previous model demand depend on consumer incomes (but no supply), supply depend on raw material prices (and not the demand)

Necessary conditions for original model: for first equation $3 = 2 + 2 - 1$ *and for the second equation* $3 > 2 + 1 - 1$ *(both equation identifiable, second overidentified)*