Advanced Econometrics

University of Warsaw

Jerzy Mycielski

Doctoral School of Social Sciences, 2024

Large sample asymptotics

- In many cases we are not able to derive the small sample distributions of estimators
- In such cases it is often derive the distributions for large samples ($n \xrightarrow{n} \infty$)
- Important concepts we are using in this context is the convergence in probability and convergence in distribution

$$a_n \xrightarrow{p} a \text{ or } plim\left(a_n\right) = a$$

$$a_n \xrightarrow{d} a$$

- Sequence of random variables a_n converge in probability if probability of a_n being close to a is converging to 1 for $n \xrightarrow{n} \infty$
- Sequence of random variables a_n converge in distribution if distribution of a_n is converging to distribution of a for $n \xrightarrow{n} \infty$

Continuous Mapping Theorem (CMT)

- CMT implies that for continuous function $g\left(\right)$

$$\text{plim} g\left(a_n\right) = g\left(\text{plim}\left(a_n\right)\right)$$

$$a_n \xrightarrow{d} a \Longrightarrow g\left(a_n\right) \xrightarrow{d} g\left(a\right)$$

- Estimators (say $\widetilde{\theta}$) are nonlinear functions of the sample
- CMT make possible the proofs of consistency of the estimators
- Generally speaking $\mathbb{E}(g\left(a_n\right)) \neq g\left(\mathbb{E}(a_n)\right)$.
- Finding stochastic limits of nonlinear functions of random variables is much easier then finding expected values and small sample distributions

Law of Large Numbers (LLN) and Central Limit Theory (CLT)

- If elements of a_n are i.i.d. and $\mathbb{E}\left(\|a_i\|\right) < \infty$
- Laws of Large Number (LLN) implies that if elements of a_n are i.i.d. and $\mathbb{E}\left(a_n\right) = \mu$ then

$$\text{plim} \frac{\sum_{i=1}^{n} a_i}{n} = \mathbb{E}\left(a_n\right) = \mu$$

- Central Limit Theorem (CLT) implies that if elements of a_n are independent and $\mathbb{E}\left(a_n\right) = \mu$ then

$$\sqrt{n}\left(\frac{\sum_{i=1}^{n} a_i}{n} - \mu\right) \xrightarrow{d} N\left(0, \mathbf{V}\right)$$

Consistency and root n consitency of estimators

- Estimator $\widehat{\theta}$ is consistent if it converges in probability to true vector of parameters θ

$$plim\left(\widehat{\theta}\right) = \theta$$

- Consistent estimator is becoming more precise when number of observations in the sample increases. In the limit the estimate is equal to true value of parameters.
- Estimator $\widehat{\theta}$ is n root consistent if

$$\sqrt{n}\left(\widehat{\theta} - \theta\right) \xrightarrow{d} N\left(0, \mathbf{V}\right)$$

- n root consistent estimator for large number has the distribution which can be approximated with normal distribution.

Properties of the MLE estimators

- MLE estimators are consistent

$$\widehat{\theta}_{MLE} \xrightarrow{d} N(0, \mathbf{V})$$

- MLE estimators are n root consistent

$$\sqrt{n}\left(\widehat{\theta}_{MLE} - \theta\right) \xrightarrow{d} N(0, \mathbf{V})$$

- The asymptotic variance covariance matrix of ML estimators is given by the inverse of the Fisher information matrix $\mathbf{V} = \mathbf{I}^{-1}(\theta)$

$$\mathbf{I}(\theta) = var\left(\frac{\partial \ell(\theta)}{\partial \theta}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\theta)}{\partial \theta' \partial \theta'}\right)$$

- Cramer-Rao theorem implies that the inverse of the information matrix $\mathbf{I}^{-1}(\theta)$ is lower limit of the variance of the consistent estimator
- Therefore the MLE estimators are asymptotically efficient

Bibliography